METRIZED SCHEDULING RELATIONS

AND APPLICATIONS

MARCH 1966

J. F. Rial

Prepared for

DEPUTY FOR ADVANCED PLANNING — DIRECTORATE OF SPECIAL SYSTEMS

ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

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ABSTRACT

The definitions of scheduling relations previously presented by the author are extended in this report to allow the translation of a relation net into an exact schedule. Time conditional conflicts in scheduling relation nets are detected and resolved by operating on extended versions of the bilateral implication and truth tables.

REVIEW AND APPROVAL

This technical report has been reviewed and is approved.

GENE D. MUNSON
Major, USAF

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SECTION I

INTRODUCTION

The analysis of logical scheduling conflicts was formulated in
(Ref. 1). Some of the relations defined (see Section 3.1 of Ref. 1),
such as K, Ψ, and Φ, exactly fix the time lines of pairs of activities
with respect to one another. A scheduling relation having this property
is _metrized_. The relations Γ, Σ, and N, for example, do not have this
property and are therefore _unmetrized_. The process of extending the
definition of an unmetrized relation so that the newly defined relation
is metrized will be called _metrization_. The metrization of the relations
defined in (Ref. 1) will be undertaken in Section II. In the same
section, we shall show that an exact schedule [1]) can be produced from a
relation net providing that (a) the relations of the net are metrized,
(b) the duration of each activity in the net is known, and (c) the start
or finish time of any of the activities is known.

In Section II we answer the question: Given the implication of a
relative product of scheduling relations, what is the metrized implica-
tion of the corresponding product of metrized relations? The reader will
recall that in (Ref. 1) a bilateral implication table was used to derive
the relation between any two activities of a relation net. Given the
same activities, along with their respective lengths, we now wish to
quantitatively express the relationship between the activities in terms
of the metrized relations of the relative strings to which the activities
belong. It will be shown in Section II that a metrized version of the
bilateral implication table of (Ref. 1) is sufficient for this task.

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1) A schedule in which the start and finish times of each activity are
fixed relative to a time line.

The truth values of cyclic relative strings were used as a basis for logical conflict detection in (Ref. 1). In Section II, we will use true cyclic strings of metrized relations to derive a basic set of conditional equations. These equations will be put to use in the detection and resolution of time conditional conflicts in Section III.

Other applications of metrized scheduling relations include the prediction of conflicts due to the delay or slipping of activity start and finish times, the extension of relation nets in a conflict-free manner (a capability crucially needed for the inter-relating of two or more profiles), and the study of the properties of alternative schedules generated by relation nets containing very general constraints such as Ω, Ω', and Δ[1]. All of these applications will be discussed in depth in this report.

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1) There are, for example, four forms of Ω', including Γ, $\breve{\Gamma}$, K, and $\breve{K}$. A relation net containing Ω will, therefore, generate at least four schedules.

METRIZED SCHEDULING RELATIONS

DEFINITION OF METRIZED SCHEDULING RELATIONS

The reader is assumed to be familiar with the algebraic defini-
tions of the scheduling relations given in (Ref. 1). As observed in the
introduction, certain of these relations are already metrized. Thus
given the durations of activities X and Y, and the start or finish
time of either X or Y, exact schedules can be produced from XKY,
XǨY, XψY, XΛY, and XΦY. We now extend the definitions of the remain-
ing relations so that under the same conditions it is possible to produce
exact schedules from any string of the form XωY. Definitions of the
metrized relations are presented in Appendix I. To produce an exact
schedule of XΓY, for example, requires not only knowledge about the
relates X and Y (durations and start or finish times), but also
more knowledge about Γ than is contained in its purely algebraic
definition. The additional parameter $i = A(*Y) - A(X*)$ is easily seen
to be the needed parameter. The metrization of Σ, on the other hand,
requires the introduction of two parameters, $i = A(*X) - A(*Y)$ and
$j = A(Y*) - A(X*)$. Each definition of Appendix I is accompanied by a
figure which is an example of the position of the relates as governed by
the defined relation.

The relations Ω, Ω', and Δ (Definitions 18, 19, and 20 of
Appendix I) require special mention, since their metrized forms can be

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expressed only as functions of previously defined relations. Observe that there are several ways of writing the metrized versions of both Ω and Δ. For example, $X\Omega Y \Longleftrightarrow X(\psi \vee B(i) \vee \breve{B}(i))Y$ and $X\Omega Y \Longleftrightarrow X(\curlywedge \vee \alpha(i) \vee \breve{\alpha}(i))Y$. In Definition 18, Ω is expressed as the union of all of the relations contained in Ω.[1] Ω' (Def. 19) is easier to handle since it is the join of 4, and only 4, mutually exclusive relations. The employment of these relations, and of their equivalent modes of expression, will be treated in the following subsection and on page 7. For now, we note that the inclusion of one or more of the relations Ω, Ω', or Δ in a relation net gives rise to a combinatorial problem of some stature.

DERIVATION OF METRIZED IMPLICATION TABLE

Denoting the metrized form of a relation ω by $M(\omega)$, and given δ as the implication of the relative product $\omega_1\omega_2$, we now wish to derive $M(\delta)$ as a function of $M(\omega_1)$ and $M(\omega_2)$. Briefly, our task is to compute the metrized implication of the relative product $M(\omega_1)\ M(\omega_2)$.

We introduce these notions by way of examples. Let $M(\omega_1) = \Gamma(i)$ and $M(\omega_2) = \Gamma(j)$. From Appendix A of (Ref. 1) we have $\Gamma\Gamma = \Gamma$. Thus $X\Gamma(i)Y\ \Gamma(j)Z = X\Gamma(k)Z$ for some value of k. From

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[1] Thus yielding a union of non-mutually exclusive relations. $X\Sigma(i,j)Y$, for instance, implies $X\alpha(j)Y$ and hence $\Sigma(i,j) \vee \alpha(j) = \alpha(j)$.

Figure 1 it is easy to see that $X\Gamma(i + ||Y|| + j)Z$ and thus

$k = i + ||Y|| + j$. The metrized implication $M(\delta)$ of $\Gamma(i)$ $\Gamma(j)$ is

thus $\Gamma(i + ||Y|| + j).$[1]

    From Appendix I of (Ref. 1) we have $\Gamma\Sigma = \breve{P}$. To metrize the

implication of $M(\Gamma)$ $M(\Sigma)$ we form the equation $\Gamma(i)$ $\Sigma(n,m) = \breve{P}(k)$[1]

and solve for K. The solution is again easily seen from Figure 2,

where, clearly, $k = i + ||Y|| + m$. Thus $\Gamma(i)$ $\Sigma(n,m) = \breve{P}(i + ||Y|| + m).$
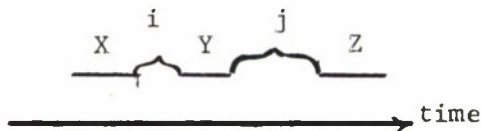

Fig. 1
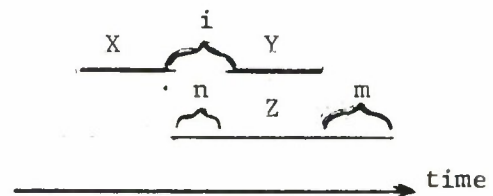
Fig. 2

    Appendix II of this paper is a complete table of metrized

implications for binary products of the relations defined in Appendix A.

The reader will observe that those implications containing more than one

term are written out in conditional form, i.e. with restrictions as to

when each form is used. As noted in the previous section, there are

various ways of writing the metrization of both Ω and Δ . The

metrization of those products having implications containing Ω and Δ

can thus take on several forms. In each of these cases we have, in

Appendix B, used what appears to be the most easily deriveable form.

$\breve{\Gamma}(n)$ $\breve{P}(m)$, for instance, equals $\breve{P}(m - (n + ||X||))$ or $\wedge$ (if

---

1) Where X, Y, and Z are understood to be, respectively, the first,
   middle, and last relates of the string.

$m = n + ||X||$) or $P(n + ||X|| - m)$, $(\Omega \vee \Omega' = \breve{P} \vee \Lambda \vee P$ since $\breve{P}$

includes Γ, K, α, and Σ, Λ includes Φ, P includes $\breve{\Gamma}$, $\breve{K}$, $\breve{\alpha}$,

and $\breve{\Sigma}$, $\breve{P} \vee \Lambda \vee P \Leftrightarrow \breve{N} \vee \psi \vee N$, and $\breve{N}$ includes β, and N includes

$\breve{\beta}$). $\breve{\Gamma}(n)$ $\breve{P}(m)$ also equals $\breve{N}(n - (m - ||Z||))$ or ψ (if $m - ||Z|| = n$)

or $N(m - ||Z|| - n)$, since, again, $\Omega \vee \Omega' = \breve{N} \vee \psi \vee N$. Either one of

the metrized implications of $\breve{\Gamma}(n)$ $\breve{P}(m)$ is thus equally acceptable,

although we have included only the former in the implication table of

Appendix II.

METRIZED VERSION OF TRUTH TABLE

The truth value of the cyclic ternary string $X\omega_1 Y\omega_2 X$ does

not, of course, change when ω_1 and ω_2 are metrized. Cyclic strings

in this section will thus always mean true cyclic strings. With the

exception of the cycles $K\breve{K}$,[1] $\psi\breve{\psi}$, and $\Lambda\Lambda$, which, in their metrized

form, carry neither new parameters nor information about the lengths

of the relates, the significance of a ternary cycle of metrized relations

is that it implies an equation holding among the parameters of the rela-

tions and the lengths of the relates. Hence, the '1' values in the

truth table of Appendix II, (Ref. 1), can each (with the above noted

exceptions) be replaced by an equation. Figures 3 and 4 exhibit the

meaning of these equations for the cycles $X\Sigma(i,j)Y$ $\breve{\alpha}(n)X$ and

$X\breve{K}Y$ $N(i)X$, respectively.
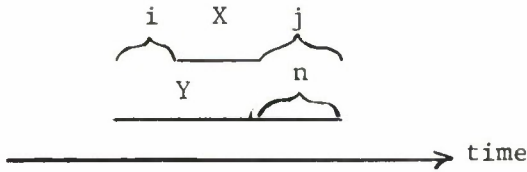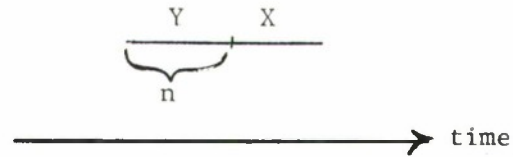
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1) Equivalently, $\breve{K}K$

Fig. 3

Fig. 4

From Figure 3 we have $i + ||X|| + n = ||Y||$. From Figure 4 we have the simpler equation $n = ||Y||$. These equations, on the surface trivial, are given in Appendix III. Their enormous role in the detection and resolution of time-conditional conflicts will be seen in Section III.

METRIZED RELATION NETS

Given $A(*X)$, $||X||$, $||Y||$, and $XM(\omega)Y$, where $M(\omega)$ is the metrization of the relation ω, it is easy to show that $A(X*)$, $A(*Y)$, and $A(Y*)$ can be computed, i.e., an exact schedule of $X\omega Y$ can be produced. For example, if $X\alpha(i)Y$, then, under the stated conditions, we have: $A(X*) = A(*X) + ||X||$, $A(Y*) = A(X*) + i$, and $A(*Y) = A(Y*) - ||Y||$. It readily follows that given a metrized relation net[1], i.e., a relation net each of whose areas is metrized, and given $||X_i||$ for each node of the net, then an exact schedule corresponding to the net can be produced providing we only know $A(*X_t)$ for some X_t.

Figure 5 shows a metrized relation net.

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1) A net with truth value equal to 1, since it makes no sense to metrize nets containing logical conflicts.

Y   ψ   Z

Γ(i)   Σ(n,m)

X   V

Γ(j)   ψ

W
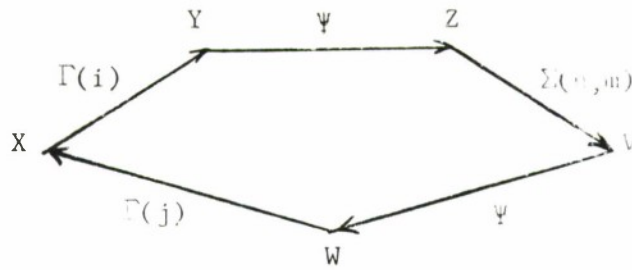
Fig. 5

Given $||X||$, $||Y||$, $||Z||$, $||V||$, $||W||$, and $A(*X)$, the schedule represented by the net is generated as follows:

1) $A(X*) = A(*X) + ||X||$,

2) $A(*Y) = A(X*) + i$,

3) $A(Y*) = A(*Y) + ||Y||$,

4) $A(*Z) = A(*Y)$,

5) $A(Z*) = A(*Z) + ||Z||$,

6) $A(*V) = A(*Z) - n$,

7) $A(V*) = A(Z*) + m$,

8) $A(*W) = A(*V)$,

9) $A(W*) = A(*X) - j$.

(observe that $||V||$ and $||W||$, although not used, could have been employed, respectively, in equations 7) and 9) in the calculations, respectively, of $A(V*)$ $(= A(*V) + ||V||)$ and $A(W*)$ $(= A(*W) + ||W||))$. These calculations do, of course, flow directly from the definitions of Appendix I.[1] The schedule is shown in Figure 6

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1) We have assumed that the durations and relation parameters are such that the relations of the net are satisfied, i.e., the net is not in conditional conflict (See Section III, page 21).

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Fig. 6

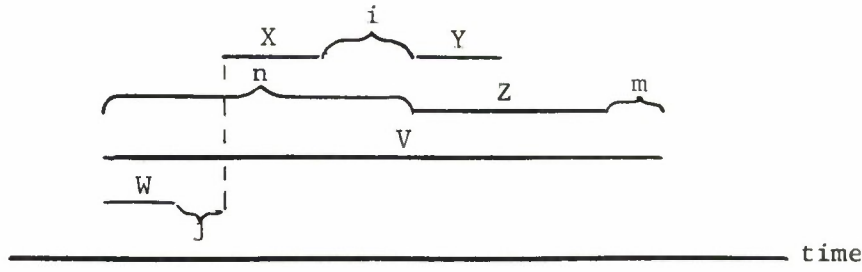It is intertesting to compute the metrized derived constraint between X and V. From X to V through Y we form the string $X\Gamma(i)Y\psi Z\Sigma(n,m)V$. From Appendix B we have $X\Gamma(i)Y\psi Z = X\Gamma(i)Z$ and $X\Gamma(i)Z\Sigma(n,m)V = X\breve{P}(i + m + ||Z||)V$, letting $n = i$, $i = n$, $j = m$, and $||Y|| = ||Z||$ in the table. From X to V through W we have $X\widetilde{\Gamma}(j)W\psi V$, which, from Appendix II, reduces to $X\breve{N}(j + ||W||)V$. Now $\breve{N} \wedge \breve{P} = \Sigma$ and hence $X\Sigma(j + ||W||, i + m + ||Z||)V,$ as seen in Figure 6.

In the initial stage of defining constraints it is sufficient, for certain activities, to express $X\Omega Y$, $X\Omega'Y$, or $X\Delta Y$ without specify-ing any ordering of X with respect to Y. There are, in fact, good reasons for employing these general relations whenever possible. The chief advantage, as will be seen from our next example (and also in our future work on the time-line packing of subschedules), is that the planner is provided with a variety of schedules for a given net, each possibly different in some critical operational aspect. It is true that the excessive use of such relations will induce combinatorial problems of great complexity. On the other hand, there is little excuse for reducing the number of solutions to a problem in the interests of achieving a simplicity and rigidity corresponding only to the most highly deterministic

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(and therefore improbable) situations.

Figure 7 shows the set of schedules generated by the metrized relation net $X\Sigma(i,j)Y$, $Y\Omega'Z$, $Z\alpha(S)W$, and $W\Lambda T$, where T denotes an interval of time with left hand endpoint at the beginning of the time line, i.e., $*T = \emptyset$ and hence $A(*T) = 0$[1]. Cases 1, 2, 3, and 4 correspond, respectively, to $\Omega' = \Gamma(n)$, $\Omega' = K$, $\Omega' = \check{\Gamma}(n)$, and $\Omega' = \check{K}$. The total durations of the subschedules are $||Y|| + n + ||Z|| + S$ (Case 1), $||Y|| + ||Z|| + S$ (Case 2), $||Z|| + n + ||Y||$ (Case 3), and $||Z|| + ||Y||$ (Case 4). If it is desired that either X or Y precede Z, then Cases 1 and 2 apply. If the shortest subschedule having either X or Y precede Z is desired, then Case 2 alone applies. The shortest subschedule is obtained in Case 4.

Case 1. $\Omega' = \Gamma(n)$

Case 2. $\Omega' = K$

Case 3. $\Omega' = \check{\Gamma}(n)$
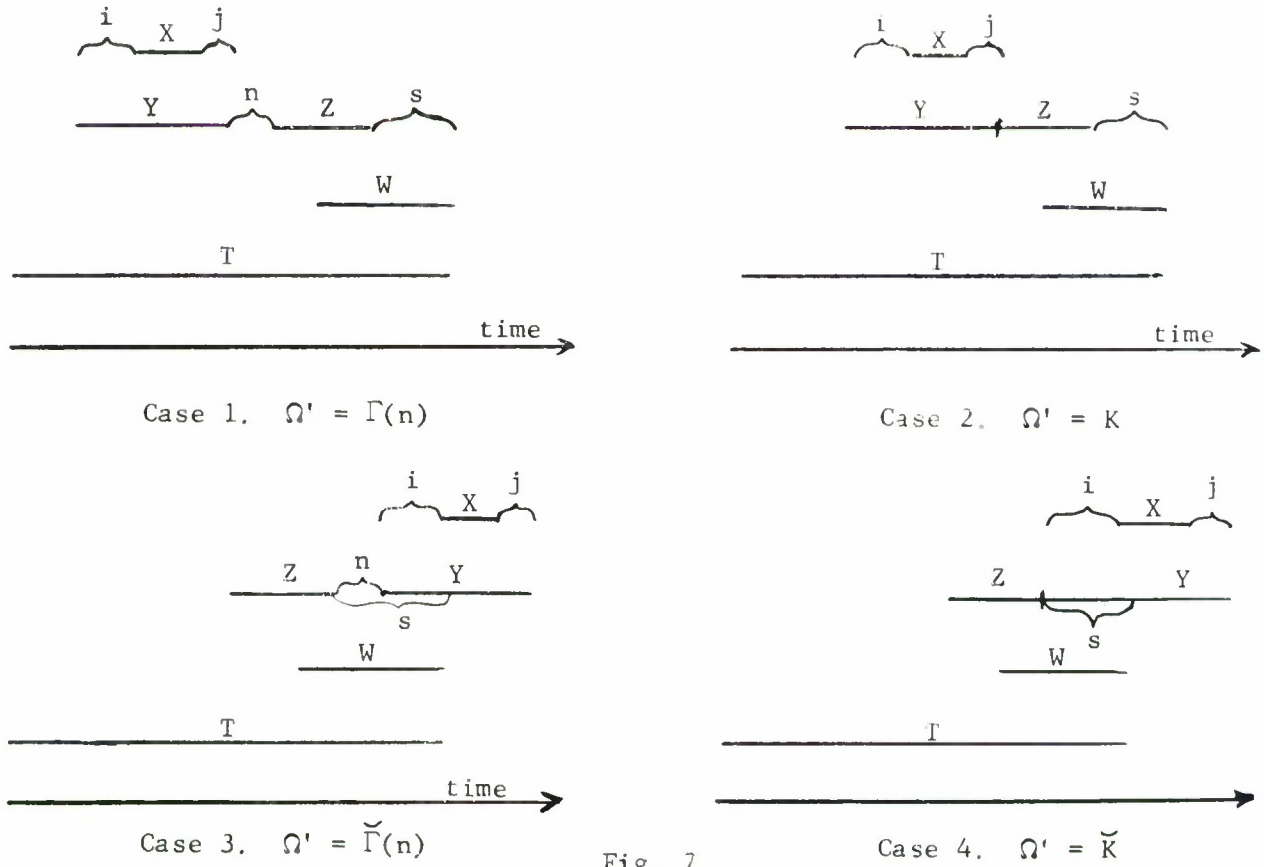
Fig. 7

Case 4. $\Omega' = \check{K}$

1) The activity T is used in defining constraints between physical activities and the schedule time line. (See Section III)

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The choice of a subschedule may also be dictated by relations holding between nets which, as soon as we know what is meant by the left and right partial complements of a net, may be defined precisely as they are between activities.

For any net n define $*n$ to be $*X_i$, $X_i \in n$, where $*X_i(N \vee \psi)*X_j$ for each $X_j \in n$. Similarly, define $n*$ to be $X*_k$, $X_k \in n$, where $X*_k(P \vee \wedge)X*_j$ for each $X_j \in n$. Finally, define $||n|| = A(n*) - A(*n)$ to be the length of the net n. Figure 8 shows the situation for a net n with seven activities.
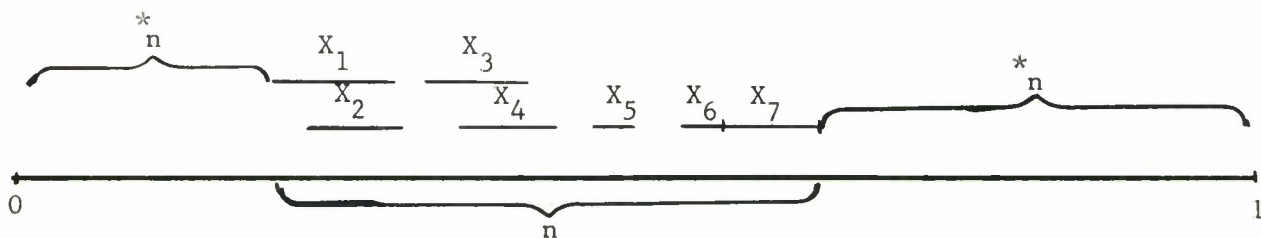
Fig. 8

Define n_1 to be net $X\Sigma Y\Omega' Z\alpha W \wedge T$[1] and let n_2 be the net $TKX'KY'KZ'$. Suppose it has been determined that $n_1\Omega'n_2$ must hold. From a Gantt chart construction it is then immediately evident that $n_1 K n_2$. We will now derive this constraint directly. Since the activity W is a part of n_1 we can write $n_1\widetilde{\Sigma}W$. Figure 9 is the relation net of the composition of n_1 and n_2.

---

1) T as in Figure 7.

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X Σ Y Ω′ Z α W ∧ Γ K X′ K Y′ K Z′

$\breve{\Sigma}$ K

n_1 Ω′ n_2

Fig. 9

From the implication table of (Ref. 1) we have $n_1(\breve{\Sigma}\wedge K)n_2 = n_1(\wedge \vee \breve{\alpha})Kn_2$
$= n_1(K\vee\breve{\beta})n_2$.   Now $\Omega' = K\vee\breve{K}\vee\Gamma\vee\breve{\Gamma}$.   Hence $n_1(K\vee\breve{\beta})\Omega'n_1$
$= n_1(K\vee\breve{\beta})(K\vee\breve{K}\vee\Gamma\vee\breve{\Gamma})n_1 = n_1(KK\vee K\breve{K}\vee K\Gamma\vee K\breve{\Gamma}\vee\breve{\beta}K\vee\breve{\beta}\breve{K}\vee\breve{\beta}\Gamma\vee\breve{\beta}\breve{\Gamma})n_1$.   From
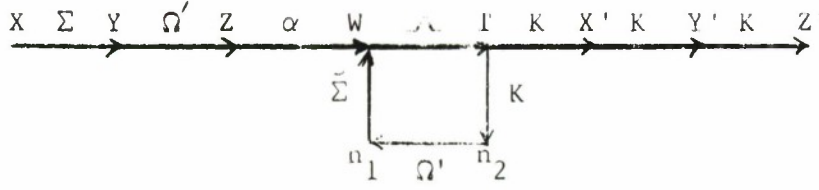the truth table of (Ref. 1) we find that all of these cyclic ternary
strings are false with the exception of $K\breve{K}$.   Hence $\Omega' = K$ and
$n_1Kn_2\breve{K}n_1$,   i.e. $n_2\breve{K}n_1$,   as we set out to prove.

It is now clear that Cases 1 and 2 are the least likely to
result in a violation of n_1Kn_2 $(||*Z|| + ||Z|| + n + ||Y||$ may exceed
$||T||$, for example, in Case 3). We shall return to this example in
Section III, where we will use metrized relations to deduce directly the
desirability of Cases 1 and 2. At this point it is important to see the
way in which relations between nets reduce the combinatorial problems
generated by relations such as Ω and Ω' within nets.

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SECTION III

CONDITIONAL CONFLICTS

BASIC FORMS FOR THE GENERATION OF CONDITIONAL EQUATIONS

Scheduling conflicts arising from logical incompatibilities among scheduling constraints were discussed in (Ref. 1). It will be recalled that a Gantt chart of a relation net exists if and only if the truth value of the net is 1. Thus given a true relation net it is always possible to satisfy the net's relations by some set of activity durations.[1] Given the start time of any activity, a Gantt chart of the net can then be produced. These statements hold true for metrized relation nets, as well, the only difference being that the set of Gantt charts corresponding to a metrized net is generally properly included in the set of Gantt charts corresponding to the unmetrized version of the same net, a consequence of the added restrictions imposed by the parameters of the metrized relations.

A metrized relation net is <u>realizable</u> if the durations of the activities, along with the start or finish time of some activity, are given. From Section II, page 7, it follows that the start and finish times of each activity of a realizable net are determined. Define the <u>metric variables</u>, or <u>m - variables</u>, of a realizable net n to be the collection $M(n)$ of durations and start times of the net's activities, along with

_____

1) The cardinality of the class of such sets is, in general, $2^{\aleph_0}$.

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the parameters of the net's relations. A realizable net is in _conditional_ _conflict_ if any of the members of M(n) undergo a change of value. We recognize four types of conditional conflict:

Type I: The m - variable changes leave each of the relations of the net fixed.

Type II: The m - variable changes induce a change of the relations of the net, but a redefinition of the m - variables restores the original relations.

Type III: The m - variable changes induce a change of the relations of the net, no restoration of the original relations is possible by redefining the m - variables, but the new net is not in logical conflict.

Type IV: Identical to Case 3 except that the new net is in logical conflict.

In this paper, we shall fully treat Types I and II, and give some analytical insight into Types III and IV.[1]

We now turn to the application of metrized relation nets to exact scheduling. The basic principles of application can be deduced from nets having only one activity that must be completed at a specified time, and we so restrict our discussion in this section. The general case, in which several activities of a net are constrained to the time line, will be treated in Section III, page 21.

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1) Types III and IV will be treated in general in "Heuristics for the Resolution of Logical and Conditional Scheduling Conflicts" by L. C. Driscoll, The MITRE Corporation. MTR-110 (To be published)

The example in Figure 7 of the previous subsection will serve as the starting point of our development of a set of equations from a realizable net. We begin with Case 1, in which $\Omega' = \Gamma(n)$. Define dX, dY, dZ, and dW, respectively, as the durations of *X, *Y, *Z, and *W.[1]

We then immediately have TN(dX)X, TN(dY)Y, TN(dZ)Z, and TN(dW)W. Figure 10 is the metrized relation net for Case 1 with the above relations included.
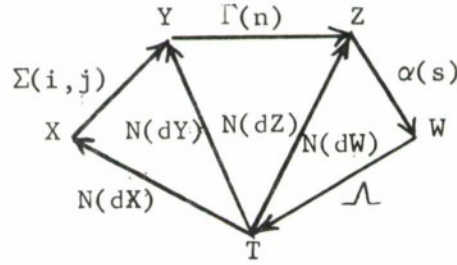
Fig. 10

From the metrized implication table of Appendix B. we find that $X\Sigma(i,j,)Y\Gamma(n)Z\alpha(s)W\wedge T = X\Gamma(j + n)Z\alpha(s)W\wedge T = X\breve{P}(j + n + s +$ $+ ||\dot{Z}||)W\wedge T = X\breve{P}(j + n + s + ||Z||)T.$ We now form the cyclic ternary string $X\breve{P}(j + n + s + ||Z||)TN(dX)X$ and, from the metrized truth table of Appendix III, find that this string is equivalent to $j + n + s + ||Z||$ $+ dX + ||X|| = ||T||$, which is clearly seen in Figure 7. The metrized implication of $Y\Gamma(n)Z\alpha(s)W\wedge T$ is $Y\breve{P}(n + s + ||Z||)T.$ From the metrized truth table $Y\breve{P}(n + s + ||Z||)TN(dY)Y$ is equivalent to

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1) Note that dX is the start time of X, etc.

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$n + \varepsilon + ||Z|| + dY + ||Y|| = ||T||$, which is again evident in Figure 7.
The equations $dZ + ||Z|| + \varepsilon = ||T||$ and $dW + ||W|| = ||T||$ can be similarly derived.

The equations derived above are called <u>partial conditional equations</u>. Each has the property of being generated from a cycle of the form $X_0\omega_0 X_1\omega_1\cdots\omega_n X_n TN(dX_0)X_0$, involving both the activity T (a section of the time line with $*T = \emptyset$), and the start time of some other activity. A <u>complete set of partial conditional equations</u> of a realizable net is a set of partial conditional equations in which each m - variable of the net occurs in some equation of the set. The completion of the set of partial equations of the net of our example is obtained by substituting $i + ||X|| + j$ for $||Y||$ in any equation containing $||Y||$, since $X\Sigma(i,j,)Y$ implies $i + ||X|| + j = ||Y||$.

A <u>total conditional equation</u> of a realizable net is the sum of any complete set of the net's partial conditional equations. A total equation of the net of our example is: $dX + dW + dZ + i + ||Y|| + ||W|| + 2(dY + ||X|| + j) + 3n + 4(\varepsilon + ||Z||) = 5||T||$. It might be supposed that so long as this equation holds there can be no violation of the relations between the activities. A quick glance at Figure 7 will convince the reader that such is not the case. An increase in n, for example, cannot be offset by a decrease in $||W||$, since, from the partial equations, $||W||$ is not a function of n. One of the uses of total conditional equations will be seen on page 19. Obviously, the

relations in a realizable net will remain fixed so long as the partial

equations hold <u>and</u> the relation parameters remain within their proper

bounds.

Let n be a realizable net, T an activity of n with

$*T = \emptyset$, and Z an activity such that $Z \wedge T$ and $*Z \neq \emptyset$. Then TNZ.

For any activity X and relation ω it is easy to verify that

$\zeta(X\omega Z \wedge TNX) = 1$. More explicitly, we have:

$$\zeta(\Sigma \wedge N) = \zeta(\check{P}N)$$

$$\zeta(\breve{\Sigma} \wedge N) = \zeta(\breve{\alpha}N)$$

$$\zeta(\Gamma \wedge N) = \zeta(\breve{P}N)$$

$$\zeta(\breve{\Gamma} \wedge N) = \zeta(\breve{\Gamma}N)$$

$$\zeta(K \wedge N) = \zeta(\breve{P}N)$$

$$\zeta(\breve{K} \wedge N) = \zeta(\breve{K}N)$$

$$\zeta(N \wedge N) = \zeta(\alpha N) \vee \zeta(\wedge N) \vee \zeta(\breve{P}N)$$

$$\zeta(\breve{N} \wedge N) = \zeta(PN) \vee \zeta(\wedge N) \vee \zeta(\breve{P}N)$$

$$\zeta(P \wedge N) = \zeta(PN)$$

$$\zeta(\breve{P} \wedge N) = \zeta(\breve{P}N)$$

$$\zeta(\alpha \wedge N) = \zeta(\breve{P}N)$$

$$\zeta(\breve{\alpha} \wedge N) = \zeta(\breve{\alpha}N)$$

$$\zeta(\beta \wedge N) = \zeta(\breve{\alpha}N) \vee \zeta(\wedge N) \vee \zeta(\breve{P}N)$$

$$\zeta(\breve{\beta} \wedge N) = \zeta(\breve{\alpha}N) \vee \zeta(\wedge N) \vee \zeta(\breve{P}N)$$

$$\zeta(\psi \wedge N) = \zeta(\beta N) \vee \zeta(\psi N)(= 0) \vee \zeta(NN)(= 0)$$

$$\zeta(\wedge \wedge N) = \zeta(\wedge N)$$

$$\zeta(\Phi \wedge N) = \zeta(\wedge N)$$

Except where indicated, all truth values in the above list are equal to 1. The list can be decomposed into the following six classes:

$$\text{Class 1 - } \breve{P}N, \; \alpha N$$

$$\text{Class 2 - } PN, \; \breve{\alpha}N$$

$$\text{Class 3 - } \breve{\Gamma}N$$

$$\text{Class 4 - } \breve{K}N$$

$$\text{Class 5 - } \Lambda N$$

$$\text{Class 6 - } \beta N \; .$$

Letting n be the parameter of the first relation, and dx the parameter of the second[1] these classes correspond (Appendix III), respectively, to the partial conditional equations:

$$1) \quad dX + ||X|| + n = ||\Gamma||$$

$$2) \quad dX + ||X|| = ||T|| + n$$

$$3) \quad dX = n + ||T||$$

$$4) \quad dX = ||T||$$

$$5) \quad dX + ||X|| = ||T||$$

$$6) \quad dX = n$$

It may appear that (6) does not involve $||\Gamma||$, but βN was the result of contracting $X\psi Z\Lambda TNX$. From the metrized implication table we find that $X\psi Z\Lambda T = X\beta(||T|| - ||Z||)T$ and hence $n = ||\Gamma|| - ||Z||$ in (6). The six classes of ternary cycles named above will be called the <u>basic forms for the Generation of Conditional Equations</u>.

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1) Thus yielding the general form $X\omega(n)TN(dX)X$.

RELATIONS BETWEEN NETS

In Section II, we presented an example of two nets, n_1, n_2, standing in the relation $n_1 K n_2$. n_1 contained the relation Ω', which gave rise to the four cases of Figure 7. It was stated that the desirability of Cases 1 and 2 could be deduced directly from a metrized relation net. A comparison of Cases 1 and 3 will illustrate the method.

The metrized derived constraint between X and T (Case 1) is $X\check{P}(j + n + s + ||z||)T$. The metrized derived constraint between X and T (Case 3) takes not one, but three forms, depending on the magnitudes of the various m - variables involved. Thus $X\Sigma(i, j)Y\Gamma(n)Z\alpha(s)W\Lambda T$ reduces to:

(i) $XP(i + n - s + ||X||)T$ if $i + n + ||X|| > s$,

(ii) $X\Lambda T$ if $i + n + ||X|| = s$,

(iii) $X\check{P}(||X|| - i - n - s)T$ if $i + n + ||X|| < s$.

Turning now to Figure 9, we substitute X for W and, in succession, P, Λ , and $\check{P}$ for Λ . The truth values of the resulting cycles are $\zeta(PK\check{K}\check{\Sigma}) = 0$, $\zeta(\Lambda K\check{K}\check{\Sigma}) = 1$, and $\zeta(\check{P}K\check{K}\check{\Sigma}) = 1$. Hence Case 3 can give rise to a logical conflict if $i + n + ||X|| > s$, whereas no such conflict is possible in Case 1. Comparisons of Case 1 with Case 4 and of Case 2 with Cases 3 and 4 will, naturally, yield similar results. This establishes the feasibility of automatically analyzing relations between nets without referring to Gantt charts.

The analysis of another problem involving relations between nets is based upon Figure 11, in which the realizable nets $n_1 = Y\Gamma(m)Z\Lambda T_1$ and $n_2 = U\Gamma(i)V \Gamma(j)W\Lambda T_2$ are related by $Y\psi U$ and $Z\Omega'V$. Suppose we

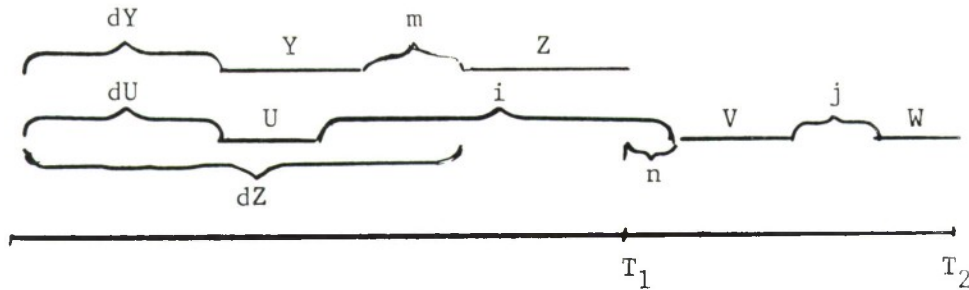let Ω' be $\Gamma(n)$, as in the figure.

Fig. 11

The total conditional equation for the combination of n_1 and n_2 is $||U|| + i + dV + dW + 2(dY + dZ + dU + n) + 3(m + ||Y||) + 4(||V|| + j) + 5(||\not{Z}|| + ||W||) = 3T_1 + 5T_2$, which yields

$n = \frac{1}{2}(5T_2 + 3T_1) - [\frac{1}{2}(||U|| + i + dV + dW) + dY + dZ + dU + 3/2(m + ||Y||) + 2(||V|| + j) + 5/2(||Z|| + ||W||)]$.

If $n > 0$ then $\Gamma(n)$ is an acceptable form of Ω'. If $n = 0$ then $\Omega' = K$. If $n < 0$ then Ω' must take the form of either $\breve{\Gamma}(n)$ or $\breve{K}$ and a new total conditional equation must be derived based upon V preceding Z. Again, if $n > 0$ then $\breve{\Gamma}(n)$ is an acceptable form of Ω'. If $n = 0$ then $\Omega' = K$. If $n < 0$ then either $Z\Omega V$ must hold or some combination of the variables dY, $||Y||$, m, dU, etc. must be assigned new values. We thus see that introducing relations between realizable nets can induce conditional conflicts within the nets and, conversely, if the m - variables of several realizable nets are fixed, then the introduction of relations between the nets is not arbitrary, but must proceed within the limitations set by the total conditional equation of the related nets.

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DETECTION AND RESOLUTION OF CONDITIONAL CONFLICTS

The partial conditional equations for Case 1, Figure 7, were derived on page 15. To increase the richness of the discussion which follows, we add the additional constraint $Y\Gamma(m)W$, thus introducing three more cycles into the net of Figure 10. The partial conditional equations for the augmented net are:

a) $dY + ||Y|| + n + ||Z|| + s = ||T||$

b) $dY + i + ||X|| + j + n + ||Z|| + s = ||T||$

c) $dX + ||X|| + j + n + ||Z|| + s = ||T||$

d) $dZ + ||Z|| + s = ||T||$

e) $dW + ||W|| = ||T||$

f) $dY + ||Y|| + m + ||W|| = ||T||$

g) $dX + ||X|| + j + m + ||W|| = ||T||$

h) $dY + i + ||X|| + j + m + ||W|| = ||T||$

Equations a) → e) were derived on page 15. f) → g) are a consequence of the added constraint $Y\Gamma(m)W$.

In this subsection, we shall explore the interdependence of m - variables in a metrized net and examine the ways in which changes in the m - variables propagate through the partial conditional equations. As stated previously, we shall confine ourselves to conditional conflicts of Types I and II.

This restriction is equivalent to the assumption that the matrix of coefficients (0's and 1's) of the linear system a) → h) is independent of m - variable changes.

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To each m -- variable V, except $||T||$, of the system

a) → h), we now add the quantity $\hat{V}$, representing the change in V.

Subtraction of the system a) → h) from the new system yields a linear

homogeneous system whose coefficient matrix is:

$$
M = \begin{array}{ccccccccccccc}
\hat{dY} & \hat{dX} & ||\hat{X}|| & ||\hat{Y}|| & \hat{i} & \hat{j} & \hat{n} & \hat{m} & \hat{dZ} & \hat{dW} & ||\hat{Z}|| & ||\hat{W}|| & \hat{s} \\
\end{array}
$$

dŶ	dX̂	‖X̂‖	‖Ŷ‖	î	ĵ	n̂	m̂	dẐ	dŴ	‖Ẑ‖	‖Ŵ‖	ŝ	
1	0	0	1	0	0	1	0	0	0	1	0	1	a
1	0	1	0	1	1	1	0	0	0	1	0	1	b
0	1	1	0	0	1	1	0	0	0	1	0	1	c
0	0	0	0	0	0	0	0	1	0	1	0	1	d
0	0	0	0	0	0	0	0	0	1	0	1	0	e
1	0	0	1	0	0	0	1	0	0	0	1	0	f
0	1	1	0	0	1	0	1	0	0	0	1	0	g
1	0	1	0	1	1	0	1	0	0	0	1	0	h

M represents a system of 8 equations and 13 unknowns. That M has

rank 6 is easily verified. From elementary algebra the nullity of M

is 7 and therefore the system has 7 linearly independent solutions form-

ing a basis for the totality of solutions of the system.

The usual method for finding the basis for the null space of

a matrix is to form the reduced echelon matrix by elementary row

operations. The reduced echelon matrix of M is:

$$E_M = \begin{pmatrix} d\hat{Y} & d\hat{X} & ||\hat{X}|| & ||\hat{Y}|| & \hat{i} & \hat{j} & \hat{n} & \hat{m} & d\hat{Z} & d\hat{W} & ||\hat{Z}|| & ||\hat{W}|| & \hat{s} \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero rows of E_M indicate that the original equations a) $\rightarrow$ h) are linearly dependent. From E_M we immediately conclude that

$$S = (-\,||\hat{Y}|| - \hat{m} - ||\hat{W}||,\ -\,||\hat{Y}|| + \hat{i} - \hat{m} - ||\hat{W}||,\ ||\hat{Y}|| - \hat{i} - \hat{j},$$
$$||\hat{Y}||,\ \hat{i},\ \hat{j},\ \hat{m} - ||\hat{Z}|| + ||\hat{W}|| - \hat{s},\ \hat{m},\ -\,||\hat{Z}|| - \hat{s},$$
$$-\,||\hat{W}||,\ ||\hat{Z}||,\ ||\hat{W}||,\ ||\hat{s}||)$$

is the general solution of the system a) $\rightarrow$ h). Equivalently, we obtain all solutions by assigning arbitrary values to $||Y||$, i, j, m, $||Z||$, $||W||$, and $||s||$, and solving the system

$$\overline{a}) \qquad d\hat{Y} + ||\hat{Y}|| + \hat{m} + ||\hat{W}|| = 0$$

$$\overline{b}) \qquad d\hat{X} + ||\hat{Y}|| - \hat{i} + \hat{m} + ||\hat{W}|| = 0$$

$$\overline{c}) \qquad ||\hat{X}|| - ||\hat{Y}|| + \hat{i} + \hat{j} = 0$$

$$\overline{d}) \qquad \hat{n} - \hat{m} + ||\hat{Z}|| - ||\hat{W}|| + \hat{s} = 0$$

$$\overline{e}) \qquad d\hat{Z} + ||\hat{Z}|| + \hat{s} = 0$$

$$\overline{f}) \qquad d\hat{W} + ||\hat{W}|| = 0$$

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for the remainder of the variables. In particular, the cases $||\hat{Y}|| = 1$,
$\hat{i} = \hat{j} = \hat{m} = ||\hat{Z}|| = ||\hat{W}|| = ||\hat{s}|| = 0;$ $i = 1, ||\hat{Y}|| = \hat{j} = \hat{m} = \dots.$
$= ||\hat{s}||;$ etc., provide a basis for the null space. Any linear combination
of basis vectors is, of course, a solution.

The basis vectors defined above allow us to develop resolutions
of conditional conflicts induced by m - variable changes within a metrized
relation net. For example, suppose in system a) → h) we allow the variable
m to decrease to m - 1[1),   i.e., $\hat{m} = -1$.   Since m is a basis variable
we set the remaining basis variables to 0,   then solve the equations
$\bar{a}) \to \bar{f})$, obtaining $\hat{dY} = 1,$ $\hat{dX} = 1,$ $\hat{n} = -1,$ and $||\hat{X}|| = \hat{dZ} = \hat{dW} = 0.$
Figure 12 shows the effects of these changes (the new variables are primed).
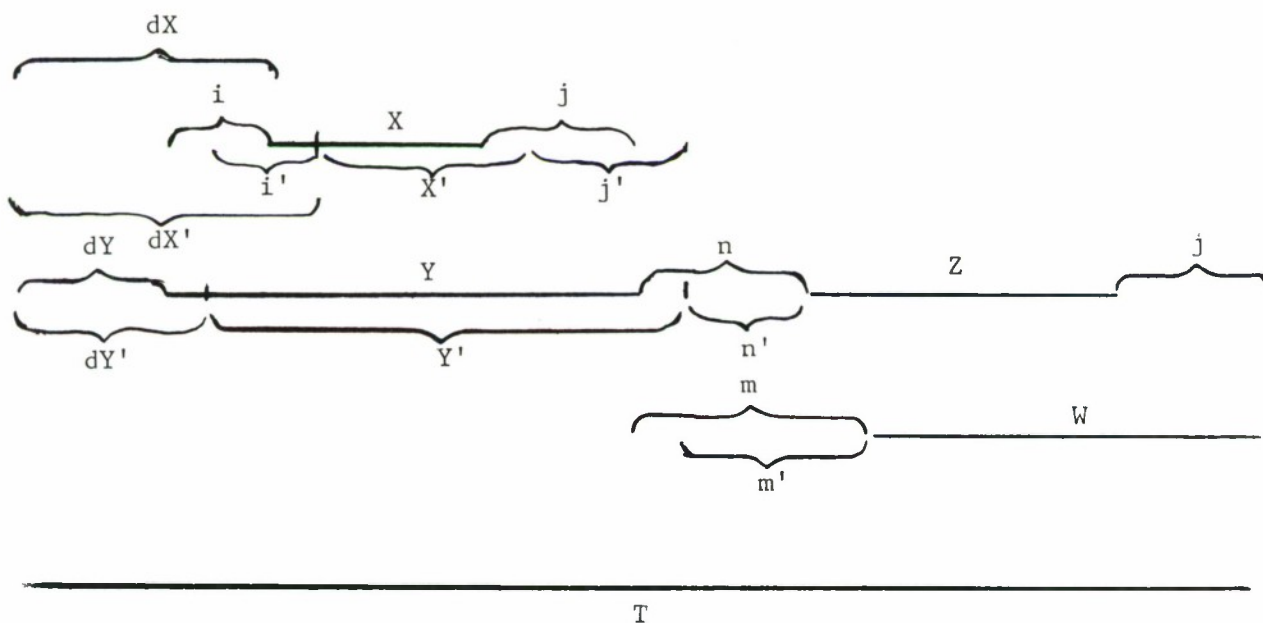

Fig. 12

The resolution above is by no means the most local, in the

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1) 1 is assumed to be a small increment of time.

sense that the change in m induces changes in the least number of the
remaining m - variables. A cursory inspection of the matrix M does,
in fact, reveal that the local resolution to a decrease (or increase) in
m is a corresponding decrease (increase) in dW and an increase (decrease)
in W.   We shall presently develop an algorithm which will, in most
cases, give local resolutions to conditional conflicts ariving from m -
variable changes.

From elementary algebra we know that a system of n homogeneous
linear equations with n unknowns has nontrivial solutions if and only
if the determinant of the system is 0.   In particular, if the rank of
the system is n, then M has only the trivial solution, which signifies
that either a Type III or Type IV conditional conflict has occurred (for
example, some combination of the coefficients of the linear system a) → h)
has been changed; equivalently, one or more of the partial conditional
equations of the net do not hold). If an m - variable V in a realizable
net is not allowed to change, then $\hat{V} = 0$, and the properties of the
linear homogeneous system corresponding to the m - variable changes are
altered.

From the partial equations a) → h) we construct a mapping
F carrying each m - variable into the set of equations to which the
variable belongs. We call F the m - variable association, or simply
the m.v.a., of the net. A few values of F are $F(dZ) = \{d\}$,
$F(||Y||) = \{a, f\}$, and $F(s) = \{a, b, c, d\}$.   In (Ref. 2), it was shown
that if B is a Boolean algebra of sets and a, b, ϵB, then the function

$$\delta(a, b) = 1 - \frac{|a \cap b|}{|a \cup b|} ,$$

where $|a \cap b|$ and $|a \cup b|$ are, respectively, the numbers of elements in

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a∩b and a∪b, is a pseudo-metric on B. We now apply this function
to define the distances between m - variables of a metrized relation
net. Thus if X and Y are m - variables of a net, then

$$\delta(X, Y) = 1 - \frac{|F(X) \cap F(Y)|}{|F(X) \cup F(Y)|} ,$$

where F is the net's m.v.a. A few distances in the system a) → h) are:

$$\delta(dZ, dW) = 1 - \frac{|\{d\} \cap \{e\}|}{|\{d, e\}|} = 1 ,$$

$$\delta(n, j) = 1 - \frac{|\{a, b, c\} \cap \{b, c, g, h\}|}{|\{a, b, c, g, h\}|} = 1 - 2/5 = .6 ,$$

$$\delta(s, ||Z||) = 1 - \frac{|\{a, b, c, d\} \cap \{a, b, c, d\}|}{|\{a, b, c, d\}|} = 1 - 1/1 = 0 .$$

The last case shows why δ is a pseudo-metric rather than a metric.

The matrix M is the representation of the m.v.a. of the net
from which the system a) → h) was derived. Using δ, we now construct
the local resolution to a change in m. Computing the distances between
m and the other m - variables we find that $||W||$ is the closest
variable to m. We next define the quasi - m - variable $m/||W||$ by
$F(m/||W||) = (F(m) \cup F(||W||)) \cap [F(m) \cap F(||W||)]'$, the symmetric difference
between $F(m)$ and $F(||W||)$. The m - variable closest to $m/||W||$
is d_Z and is fact $\delta(dW, m/||W||) = 0$. The computation ends here and
it is evident that an increase (decrease) in m can be accompanied by an
equivalent decrease (increase) in $||W||$, which will in turn induce an
equal increase (decrease) in dW. Observe that this resolution, although
local, is an equal increment resolution, and not necessarily the most
desirable. The existence of a non-trivial resolution still depends, of

course, on the rank and structure of M, and the formation of E_M still yields the most general resolutions.

We now extend the above process of conditional conflict resolution to nets containing two or more activities, each of which must be completed at a specified time. The method will be adequately demonstrated through Figures 13a) and 13b) which show, respectively, a metrized relation net and its corresponding schedule. Activities Z and R finish, respectively, at times T_1 and T_2.
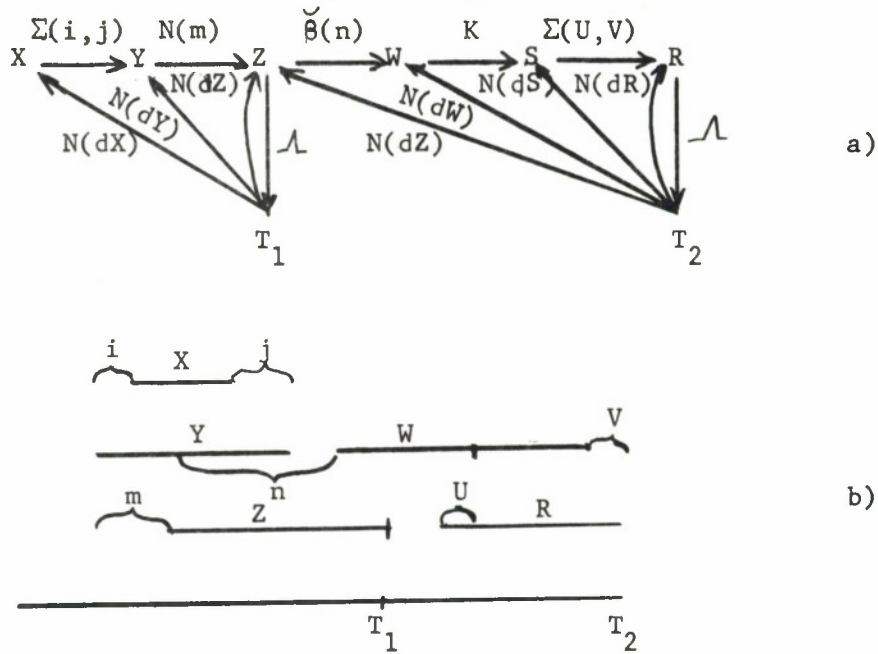

Fig. 13

There are several ways of generating the partial conditional equations from the net of Figure 13a). We may, for example, generate the equations first with respect to the cycles containing T_1, and then with respect to the cycles containing T_2, the remaining equations coming from $X\Sigma(i, j)Y$ and $s\Sigma(U, V)R$, yielding, respectively, $i + ||X|| + j = ||Y||$ and $U + ||S|| + V = ||R||$. On the other hand we observe that $||T_1|| \leq ||T_2|| \Rightarrow T_2 \breve{\alpha}(||T_2|| - ||T_1||)T_1$, the derived constraint

between T_2 and T_1. Figure 14 shows an alternative form of the net of Figure 13a).
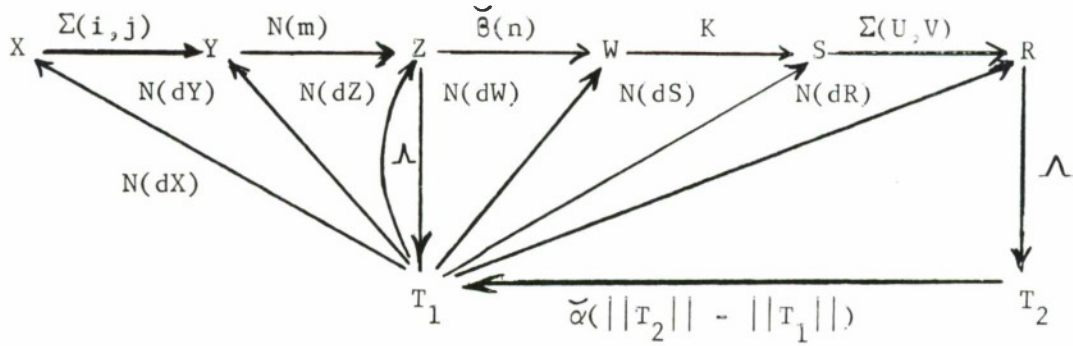

Fig. 14

We may now generate the partial conditional equations with respect to the cycles containing both T_2 and T_1 (through $N(dX)$, $N(dY)$,...., $N(dR)$). Finally we may generate first with respect to cycles containing only T_1 (through $N(dX)$, $N(dY)$, $N(dZ)$) and then with respect to the cycles containing T_2 and T_1 (through $N(dW)$, $N(dS)$, $N(dR)$). The first method requires the least computation, but the symmetry of the second method is also attractive. The occurrence of a number of time-bound activities in a metrized relation net does not, at any rate, introduce any new problems in generating the partial conditional equations for the net.

We conclude this section with a discussion of dynamic conditional conflict detection and resolution (again restricting ourselves to Type I and Type II conflicts). The discussion will be based on the example used in the beginning of this section. Denote the start time of any activity,

or relation parameter, V , by $A(*V)$. In general the start times $A(*V_j)$ of the activities and relation parameters V_j of a realizable net are partially ordered. In Case 1 of Figure 7 (with $Y\Gamma(m)W$ added) we thus have $A(*Y) = A(*i) \leq A(*X) < A(*j) \leq A(*n) = A(*m) < A(*Z)$ $\leq A(*W) < A(*s) < A(*T)$, if we assume $||W|| \leq ||Z|| + s$.

Suppose that the activities of the net are actually being performed. The first activity to start is Y and suppose, for the sake of argument, that Y is started late. From the m - variable change matrix M we form E_M and obtain a new set of values for some subset of the variables. It is now clear that the next change in the m - variables cannot involve $\hat{dY}$ and hence $\hat{dY} = 0$. The column in M , and hence E_M , corresponding to $\hat{dY}$ is thus set to 0. Let M_1 and E_{M_1} be the new matrices thus formed. We infer that as the schedule is being completed we must form an ordered set of m - variable change matrices M_i , each having one less non-zero column than its predecessor. The ordering of the set of matrices exactly corresponds to the time ordering of the start and finish times of the net's activities. If at any time we encounter a nXn matrix M_K with rank n , then only the trivial solution obtains and either a Type III or Type IV conflict has been encountered. If no such matrix is encountered then the schedule can be realized with no change of relations, providing that only positive values of the m - variables are generated as solutions to conditional conflicts.

CONCLUSIONS

The detection and resolution of both logical and conditional conflicts in a complex plan, e.g. MOL, will most certainly require a partitioning of the plan into manageable sections, and thus induce partitions of the

schedules corresponding to that plan. Some of the problems of defining relations between subschedules have been briefly explored in this paper. In a subsequent paper we shall treat this subject in depth, with the aim of discovering rules for partitioning schedules, and of developing a full analysis of the problem of defining relations between subschedules.
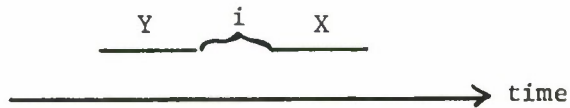
APPENDIX I

METRIZED SCHEDULING RELATIONS[1]

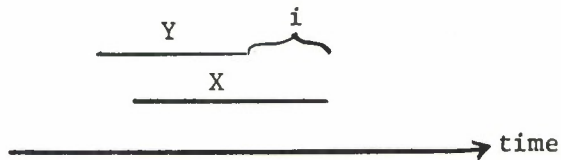(A(*X) and A(X*) are, respectively, the start and finish times of activity X).

<u>Def. 1</u> $X\Gamma(i)Y \Leftrightarrow A(X*) + i = A(*Y), \quad i > 0$
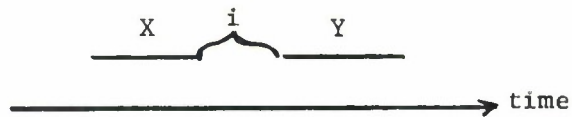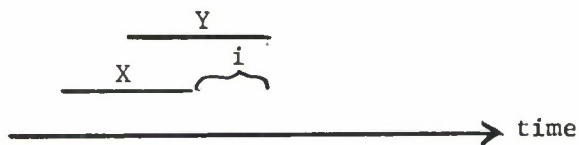

<u>Def. 2</u> $X\breve{\Gamma}(i)Y \Leftrightarrow A(*X) = A(Y*) + i, \quad i > 0$

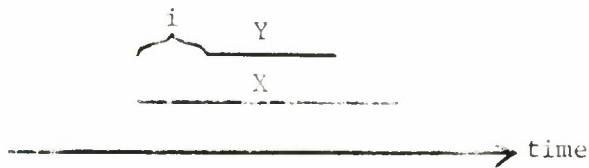<u>Def. 3</u> $XP(i)Y \Leftrightarrow A(Y*) + i = A(X*), \quad i > 0$

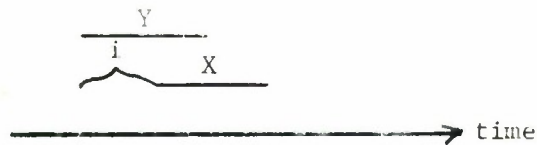<u>Def. 4</u> $X\breve{P}(i)Y \Leftrightarrow A(Y*) = A(X*) + i, \quad i > 0$

---

1) As observed in the Introduction, K, K, ψ, Λ, and Φ are metrized without additional parameters. They are included here only in the interest of completeness.

<u>Def. 5</u> $XN(i)Y <=> A(*X) + i = A(*Y), \quad i > 0$

<u>Def. 6</u> $X\breve{N}(i)Y <=> A(*X) = A(*Y) + i, \quad i > 0$

<u>Def. 7</u> $X\alpha(i)Y <=> A(Y*) = A(X*) + i, \quad 0 < i < ||Y||$
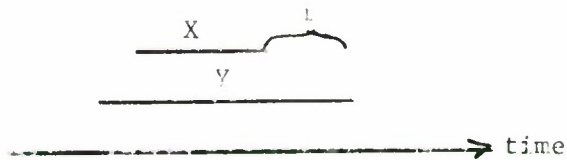
<u>Def. 8</u> $X\breve{\alpha}(i)Y <=> A(Y*) + i = A(X*), \quad 0 < i < ||X||$

<u>Def. 9</u> $X\beta(i)Y <=> A(*Y) + i = A(*X), \quad 0 < i < ||Y||$

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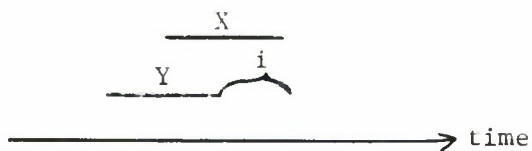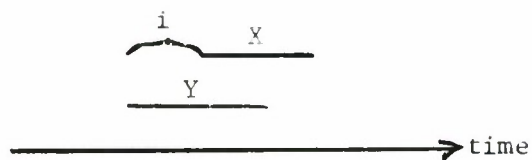
<u>Def. 10</u> $X\breve{\beta}(i)Y \Longleftrightarrow A(*X) + i = A(*Y), \quad 0 < i < ||X||$
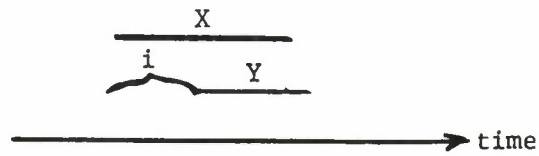
<u>Def. 11</u> $X\Sigma(i, j)Y \Longleftrightarrow A(*Y) + i = A(*X) \quad$ and

$$A(Y*) = A(X*) + j, \quad i \geq 0, \; j \geq 0, \; i + j > 0$$
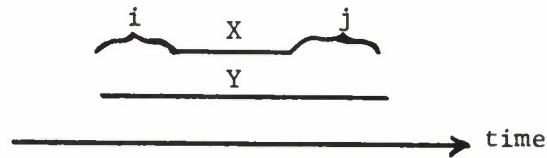
<u>Def. 12</u> $X\breve{\Sigma}(i, j)Y \Longleftrightarrow A(*X) + i = A(*Y) \quad$ and

$$A(Y*) + j = A(X*), \quad i \geq 0, \; j \geq 0, \; i + j > 0$$
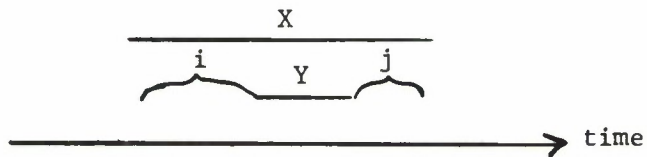
<u>Def. 13</u> $XKY \Longleftrightarrow A(X*) = A(*Y)$
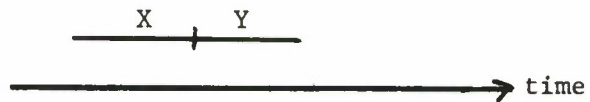
<u>Def. 14</u> $X\breve{K}Y \Longleftrightarrow A(Y*) = A(*X)$

<u>Def. 15</u> $X\psi Y \Leftrightarrow A(*X) = A(*Y)$

$\qquad$ time

<u>Def. 16</u> $X \wedge Y \Leftrightarrow A(X*) = A(Y*)$

$\qquad$ time
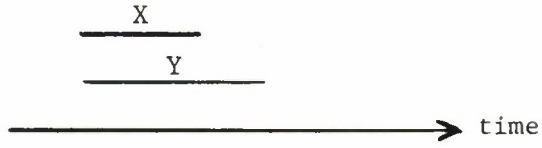
<u>Def. 17</u> $X \Phi Y \Leftrightarrow A(*X) = A(*Y)$ and $A(X*) = A(Y*)$

$\qquad$ time

<u>Def. 18</u> $X \Omega Y \Leftrightarrow X(\psi \vee \beta(i) \vee \breve{\beta}(i) \vee \wedge \vee \alpha(i) \vee \breve{\alpha}(i) \vee \Phi \vee \Sigma(i,j) \vee \breve{\Sigma}(i,j))Y$

<u>Def. 19</u> $X \acute{\Omega} Y \Leftrightarrow X(\Gamma(i) \vee K \vee \breve{K} \vee \breve{\Gamma}(i))Y$

<u>Def. 20</u> $X \Delta Y \Leftrightarrow X(\psi \vee \beta(i) \vee \breve{\beta}(i) \vee \wedge \vee \alpha(i) \vee \breve{\alpha}(i) \vee \Phi \vee \Sigma(i,j) \vee$
$$\breve{\Sigma}(i,j) \vee K \vee \breve{K})Y$$

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APPENDIX II

METRIZED IMPLICATION TABLE

The metrized implications of ternary strings of the relations defined in Appendix I are given below. The general form is $X\omega(i)Y\overline{\omega}(j)Z$ $= X\omega_1(i_1)Z \vee X\omega_2(i_2)Z \vee \cdots \vee X\omega_n(i_n)Z.$ The relates are not included in the table and X, Y, and Z are always to be interpreted, respectively, as the initial, intermediate, and final relates of the string. To avoid long, cumbersome, expressions joins in implications are represented by commas, thus yielding lists of the elements of implications. The table is divided into relation classes, the first factor in each product being the relation class to which the product belongs. The products in each relation class correspond to a complete row of the bilateral implication table of (Ref. 1). The employment of the forms of Ω, Ω', and Δ is explained in Section II, page 4.

$\underline{\Sigma(i,j)}$

$$\Sigma(i,j)\Sigma(n,m) = \Sigma(i+n,\ j+m)$$

$$\Sigma(i,j)\breve{\Sigma}(n,m) = \breve{N}(i-n),$$
$$= \psi \text{ if } i = n,$$
$$= N(n-i).$$

$$\Sigma(i,j)\Gamma(n) = \Gamma(j+n)$$

$$\Sigma(i,j)\breve{\Gamma}(n) = \breve{\Gamma}(i+n)$$

$$\Sigma(i,j)K = \Gamma(j)$$

$$\Sigma(i,j)\breve{K} = \breve{\Gamma}(j)$$

$$\Sigma(i,j)N(n) = \breve{N}(i-n),$$
$$= \psi \text{ if } i = n,$$
$$= N(n-i).$$

$$\Sigma(i,j)\breve{N}(n) = \breve{N}(n+i)$$

$$\Sigma(i,j)P(n) = P(n-j),$$
$$= \Lambda \text{ if } j = n,$$
$$= \breve{P}(j-n).$$

$$\Sigma(i,j)\breve{P}(n) = \breve{P}(n+j)$$

$$\Sigma(i,j)\alpha(n) = \breve{P}(n+j)$$

$$\Sigma(i,j)\breve{\alpha}(n) = P(n-j),$$
$$= \Lambda \text{ if } n = j,$$
$$= \breve{P}(j-n).$$

$$\Sigma(i,j)\beta(n) = \breve{N}(i+n)$$

$$\Sigma(i,j)\breve{\beta}(n) = \breve{N}(i-n),$$
$$= \psi \text{ if } i = n,$$
$$= N(n-i).$$

$$\Sigma(i,j)\psi = \breve{N}(i)$$

$$\Sigma(i,j)\Lambda = \breve{P}(j)$$

$\underline{\breve{\Sigma}(i,j)}$

$$\breve{\Sigma}(i,j)\Sigma(n,m) = \beta(n-i),$$
$$= \psi \text{ if } n = i,$$
$$= \breve{\beta}(i-n).$$

$$\breve{\Sigma}(i,j)\breve{\Sigma}(n,m) = \breve{\Sigma}(i+n,\ j+m)$$

$$\breve{\Sigma}(i,j)\Gamma(n) = N(i+n+||Y||)$$

$$\breve{\Sigma}(i,j)\breve{\Gamma}(n) = P(j+n+||Y||)$$

$$\breve{\Sigma}(i,j)K = \breve{\beta}(i+||Y||)$$

$$\breve{\Sigma}(i,j)N(n) = N(i+n),$$
$$= \Gamma(n-(j+||Y||))$$

$$\breve{\Sigma}(i,j)\breve{N}(n) = \breve{N}(n-i),$$
$$= \psi \text{ if } n = i,$$
$$= \breve{\beta}(i-n).$$

$$\breve{\Sigma}(i,j)P(n) = P(j+n)$$

$$\breve{\Sigma}(i,j)\breve{P}(n) = \Gamma(n-||Z||-j),$$
$$= K \text{ if } n=||Z||+j,$$
$$= \alpha(n-j).$$

$$\breve{\Sigma}(i,j)\alpha(n) = \breve{\alpha}(j-n),$$
$$= \Lambda \text{ if } j = n,$$
$$= \alpha(n-j)$$

$$\breve{\Sigma}(i,j)\breve{\alpha}(n) = \breve{\alpha}(n+j)$$

$$\breve{\Sigma}(i,j)\beta(n) = \breve{\beta}(i-n),$$
$$= \psi \text{ if } i = n,$$
$$= \beta(n-i).$$

$$\breve{\Sigma}(i,j)\breve{\beta}(n) = \breve{\beta}(n+i)$$

$$\breve{\Sigma}(i,j)\psi = N(i)$$

$$\breve{\Sigma}(i,j)\Lambda = \breve{\alpha}(j)$$

$$\breve{\Sigma}(i,j)\breve{K} = \breve{\alpha}(j+||Y||)$$

$$\underline{\Gamma(n)} \qquad \Gamma(n)\Sigma(i,j) = \breve{P}(n+j+||Y||)$$

$$\Gamma(n)\breve{\Sigma}(i,j) = \Gamma(n+i)$$

$$\Gamma(n)\Gamma(m) = \Gamma(n+||Y||+m)$$

$$\Gamma(n)\breve{\Gamma}(m) = P(m-n),$$

$$= \Lambda \text{ if } m = n,$$

$$= \breve{P}(n-m)$$

$$\Gamma(n)K = \Gamma(n+||Y||)$$

$$\Gamma(n)\breve{K} = \breve{P}(n)$$

$$\Gamma(n)N(m) = N(n+m)$$

$$\Gamma(n)\breve{N}(m) = P(m-||Z||-n),$$

$$= \Lambda \text{ if } m-||Z|| = n,$$

$$= \breve{P}(n-(m-||Z||)).$$

$$\Gamma(n)P(m) = P(m-(n+||Y||)),$$

$$= \Lambda \text{ if } m = n+||Y||,$$

$$= \breve{P}(n+||Y||-m).$$

$$\Gamma(n)\breve{P}(m) = \breve{P}(n+m+||Y||)$$

$$\Gamma(n)\alpha(m) = \breve{P}(n+m+||Y||)$$

$$\Gamma(n)\breve{\alpha}(m) = \breve{P}(n+(||Y||-m))$$

$$\Gamma(n)\beta(m) = \breve{P}(n+(||Z||-m))$$

$$\Gamma(n)\breve{\beta}(m) = \Gamma(n+m)$$

$$\Gamma(n)\psi = \Gamma(n)$$

$$\Gamma(n)\Lambda = \breve{P}(n+||Y||)$$

$$\underline{\breve{\Gamma}(n)} \qquad \breve{\Gamma}(n)\Sigma(i,j) = \breve{P}(i+||Y||+n)$$

$$\breve{\Gamma}(n)\breve{\Sigma}(i,j) = \breve{\Gamma}(j+n)$$

$$\breve{\Gamma}(n)\Gamma(m) = \breve{N}(n-m),$$

$$= \psi \text{ if } n = m,$$

$$= N(m-n).$$

$$\breve{\Gamma}(n)\breve{\Gamma}(m) = \breve{\Gamma}(n+||Y||+m)$$

$$\breve{\Gamma}(n)K = \breve{N}(n)$$

$$\breve{\Gamma}(n)\breve{K} = \breve{\Gamma}(n+||Y||)$$

$$\breve{\Gamma}(n)N(m) = \breve{N}(n+||Y||-m),$$

$$= \psi \text{ if } ||Y||+n = m,$$

$$= N(m-(n+||Y||)).$$

$$\breve{\Gamma}(n)\breve{N}(m) = \breve{N}(n+m+||Y||)$$

$$\breve{\Gamma}(n)P(m) = \breve{\Gamma}(n+m)$$

$$\breve{\Gamma}(n)\breve{P}(m) = \breve{P}(m-(n+||X||)),$$

$$= \Lambda \text{ if } m = n+||X||,$$

$$= P(n+||X||-m).$$

$$\breve{\Gamma}(n)\alpha(m) = \breve{N}(n+||Z||-m)$$

$$\breve{\Gamma}(n)\breve{\alpha}(m) = \breve{\Gamma}(n+m)$$

$$\breve{\Gamma}(n)\beta(m) = \breve{N}(m+||Y||+n)$$

$$\breve{\Gamma}(n)\breve{\beta}(m) = \breve{N}(||Y||-m+n)$$

$$\breve{\Gamma}(n)\psi = \breve{N}(||Y||+n)$$

$$\breve{\Gamma}(n)\Lambda = \breve{\Gamma}(n)$$

$\underline{K}$

$$K\Sigma(i,j) = \alpha(||Y||+j)$$
$$K\tilde{\Sigma}(i,j) = \Gamma(i)$$
$$K\Gamma(n) = \Gamma(n+||Y||)$$
$$K\tilde{\Gamma}(n) = P(n)$$
$$KK = \Gamma(||Y||)$$
$$K\breve{K} = \Lambda$$
$$KN(n) = \Gamma(n)$$
$$K\tilde{N}(n) = \tilde{N}(n-||X||),$$
$$= \psi \text{ if } n = ||X||,$$
$$= N(||X||-n).$$
$$KP(n) = \breve{P}(||Y||-n),$$
$$= \Lambda \text{ if } ||Y|| = n,$$
$$= P(n-||Y||).$$
$$K\tilde{P}(n) = \tilde{P}(||Y||+n)$$
$$K\alpha(n) = \breve{P}(||Y||+n)$$
$$K\tilde{\alpha}(n) = \breve{P}(||Y||-n)$$
$$K\beta(n) = \alpha(||Z||-n)$$
$$K\tilde{\beta}(n) = \Gamma(n)$$
$$K\psi = K$$
$$K\Lambda = \breve{P}(||Y||)$$

$\breve{K}$

$$\breve{K}\Sigma(i,j) = \beta(i+||Y||)$$
$$\breve{K}\tilde{\Sigma}(i,j) = \breve{\Gamma}(j)$$
$$\breve{K}\Gamma(n) = N(n)$$
$$\breve{K}\tilde{\Gamma}(n) = \breve{\Gamma}(||Y||+n)$$
$$\breve{K}K = \psi$$
$$\breve{K}\breve{K} = \breve{\Gamma}(||Y||)$$
$$\breve{K}N(n) = \tilde{N}(||Y||-n),$$
$$= \psi \text{ if } n = ||Y||,$$
$$= N(n-||Y||).$$
$$\breve{K}\tilde{N}(n) = \tilde{N}(n+||Y||)$$
$$\breve{K}P(n) = \breve{\Gamma}(n)$$
$$\breve{K}\tilde{P}(n) = P(||X||-n),$$
$$= \Lambda \text{ if } ||X|| = n,$$
$$= \breve{P}(n-||X||).$$
$$\breve{K}\alpha(n) = \beta(||Z||-n)$$
$$\breve{K}\tilde{\alpha}(n) = \breve{\Gamma}(n)$$
$$\breve{K}\beta(n) = \tilde{N}(n+||Y||)$$
$$\breve{K}\tilde{\beta}(n) = \beta(||Y||-n)$$
$$\breve{K}\psi = \tilde{N}(||Y||)$$
$$\breve{K}\Lambda = \breve{K}$$

$\underline{N(n)}$ $N(n)\Sigma(i,j)$ = $\theta(i-n)$, $N(n)\breve{\theta}(m)$ = $N(m+n)$

                                = ψ if i = n, $N(n)\psi$ = $N(n)$

                                = $N(n-i)$. $N(n)\Lambda$ = $\alpha(||X||-(||Y||+n))$,

         $N(n)\breve{\Sigma}(i,j)$ = $N(n+i)$ = Λ if $||X||$ = $||Y||+n$,

         $N(n)\Gamma(m)$ = $N(n+||Y||+m)$ = $\breve{P}(||Y||+n-||X||)$.

         $N(n)\breve{\Gamma}(m)$ = $\breve{N}(||Z||+m-n)$,

                                = ψ if n = $||Z||+m$,

                                = $N(n-(||Z||+m))$.

         $N(n)N(m)$ = $N(n+m)$

         $N(n)\breve{N}(m)$ = $\breve{N}(m-n)$,

                                = ψ if n = m,

                                = $N(n-m)$.

         $N(n)P(m)$ = $\breve{N}(||Z||+m-(n+||Y||))$,

                                = ψ if $||Z||+m$ = $n+||Y||$,

                                = $N(n+||Y||-(||Z||+m))$.

         $N(n)\breve{P}(m)$ = $\breve{\alpha}(||X||-(n+||Y||+m))$,

                                = Λ if $||X||$ = $n+||Y||+m$,

                                = $\breve{P}(n+||Y||+m-||X||)$.

         $N(n)\alpha(m)$ = $\breve{\alpha}(||X||-(n+||Y||+m))$,

                                = Λ if $n+||Y||+m$ = $||X||$,

                                = $\breve{P}(n+||Y||+m-||X||)$.

         $N(n)\breve{\alpha}(m)$ = $\breve{\alpha}(m-(||Y||+n-||X||))$,

                                = Λ if $||Y||+n-||X||$ = m,

                                = $\breve{P}(||Y||+n-||X||-m)$.

         $N(n)\theta(m)$ = $\theta(m-n)$,

                                = ψ if m = n.

                                = $N(n-m)$.

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$\underline{\widetilde{N}(n)}$ $\widetilde{N}(n)\Sigma(i,j) = \widetilde{N}(i+n)$ $\widetilde{N}(n)\beta(m) = \widetilde{N}(n+m)$

$\widetilde{N}(n)\breve{\Sigma}(i,j) = \widetilde{N}(n-i)$ $\widetilde{N}(n)\breve{\beta}(m) = \widetilde{N}(n-m)$,

$\qquad\qquad\qquad = \psi$ if $i = n$, $= \psi$ if $n = m$,

$\qquad\qquad\qquad = N(i-n)$ $= N(m-n)$.

$\widetilde{N}(n)\Gamma(m) = N(||Y||-n+m)$, $\widetilde{N}(n)\psi = \widetilde{N}(n)$

$\qquad\qquad = \psi$ if $||Y||+m = n$, $\widetilde{N}(n)\Lambda = P(n+||X||-||Y||)$,

$\qquad\qquad = \widetilde{N}(n-(||Y||+m))$. $= \Lambda$ if $n+||X|| = ||Y||$,

$\widetilde{N}(n)\breve{\Gamma}(m) = \breve{\Gamma}(m+n)$ $= \breve{P}(||Y||-(n+||X||))$.

$\widetilde{N}(n)K = \widetilde{N}(n-||Y||)$,

$\qquad\quad = \psi$ if $n = ||Y||$,

$\qquad\quad = N(||Y||-n)$.

$\widetilde{N}(n)\breve{K} = \breve{\Gamma}(n)$

$\widetilde{N}(n)N(m) = \widetilde{N}(n-m)$,

$\qquad\qquad = \psi$ if $n = m$,

$\qquad\qquad = N(m-n)$.

$\widetilde{N}(n)\widetilde{N}(m) = \widetilde{N}(n+m)$

$\widetilde{N}(n)P(m) = \widetilde{N}(||Z||+m+n-||Y||)$,

$\qquad\qquad = \psi$ if $||Z||+m+n = ||Y||$,

$\qquad\qquad = N(||Y||-(n+m+||Z||))$.

$\widetilde{N}(n)\breve{P}(m) = \widetilde{N}(n-(||Y||+m-||Z||))$,

$\qquad\qquad = \psi$ if $n = ||Y||+m-||Z||$,

$\qquad\qquad = N(||Y||+m-||Z||-n$.

$\widetilde{N}(n)\alpha(m) = \widetilde{N}(n-(||Y||+m-||Z||))$,

$\qquad\qquad = \psi$ if $n = ||Y||+m-||Z||$,

$\qquad\qquad = N(||Y||+m-||Z||-n)$.

$\widetilde{N}(n)\breve{\alpha}(m) = \widetilde{N}(||Z||+m+n-||Y||)$,

$\qquad\qquad = \psi$ if $||Z||+m+n = ||Y||$,

$\qquad\qquad = N(||Y||-(n+m+||Z||))$.

$\underline{P(n)}$

$P(n)\Sigma(i,j) = P(n-j),$

$\qquad\qquad = \Lambda$ if $n = j,$

$\qquad\qquad = \alpha(j-n).$

$P(n)\breve{\Sigma}(i,j) = P(n+j)$

$P(n)\Gamma(m) \quad = \alpha(||Z||+m-n)$ if $n>m,$

$\qquad\qquad = K$ if $m = n,$

$\qquad\qquad = \Gamma(||Z||+m-n)$ if $m>n,$

$P(n)\breve{\Gamma}(m) \quad = P(m+||Y||+n)$

$P(n)K \qquad = \breve{N}(n-||X||),$

$\qquad\qquad = \psi$ if $n = ||X||,$

$\qquad\qquad = \breve{\beta}(||X||-n).$

$P(n)\breve{K} \qquad = P(||Y||+n)$

$P(n)N(m) \quad = \breve{N}(||Y||+n-(m+||X||)),$

$\qquad\qquad = \psi$ if $||Y||+n = m+||X||,$

$\qquad\qquad = N(m+||X||-(||Y||+n)).$

$P(n)\breve{N}(m) \quad = \breve{N}(m+||Y||+n-||X||),$

$\qquad\qquad = \psi$ if $m+||Y||+n = ||X||,$

$\qquad\qquad = \breve{\beta}(||X||-(m+||Y||+n)).$

$P(n)P(m) \quad = P(n+m)$

$P(n)\breve{P}(m) \quad = P(n-m),$

$\qquad\qquad = \Lambda$ if $n = m,$

$\qquad\qquad = \breve{P}(m-n).$

$P(n)\alpha(m) \quad = P(n-m),$

$\qquad\qquad = \Lambda$ if $n = m,$

$\qquad\qquad = \alpha(m-n).$

$P(n)\breve{\alpha}(m) \quad = P(n+m)$

$P(n)\beta(m) = \breve{N}(m+||Y||+n-||X||),$

$\qquad\qquad = \psi$ if $m+||Y||+n = ||X||,$

$\qquad\qquad = \breve{\beta}(||X||-(m+||Y||+n)).$

$P(n)\breve{\beta}(m) = \breve{N}(||Y||+n-(||X||+m)),$

$\qquad\qquad = \psi$ if $||Y||+n = ||X||+m,$

$\qquad\qquad = \breve{\beta}(||X||+m-(||Y||+n)).$

$P(n)\psi \qquad = \breve{N}(||Y||+n-||X||),$

$\qquad\qquad = \psi$ if $||Y||+n = ||X||,$

$\qquad\qquad = \breve{\beta}(||X||-(||Y||+n)).$

$P(n)\Lambda \qquad = P(n)$

$\underline{\breve{P}(n)}$

$\breve{P}(n)\Sigma(i,j) = \breve{P}(n+j)$

$\breve{P}(n)\breve{\Sigma}(i,j) = P(j-n),$

$\qquad = \Lambda \text{ if } j = n,$

$\qquad = \breve{P}(n-j).$

$\breve{P}(n)\Gamma(m) = \Gamma(n+m)$

$\breve{P}(n)\breve{\Gamma}(m) = \breve{N}(||Z||+m+||Y||-(||X||+n)),$

$\qquad = \psi \text{ if } ||Z||+m+||Y|| = ||X||+n,$

$\qquad = N(||X||+n-(||Z||+m+||Y||)).$

$\breve{P}(n)K = \Gamma(n)$

$\breve{P}(n)\breve{K} = \breve{N}(||Z||+||Y||-(||X||+n)),$

$\qquad = \psi \text{ if } ||Z||+||Y|| = ||X||+n,$

$\qquad = \acute{N}(||X||+n-(||Z||+||Y||)).$

$\breve{P}(n)N(m) = \breve{N}(||Y||-(||X||+n+m)),$

$\qquad = \psi \text{ if } ||Y|| = ||X||+n+m,$

$\qquad = N((||X||+n+m)-||Y||).$

$\breve{P}(n)\breve{N}(m) = \breve{N}(||Y||+m-(||X||+n)),$

$\qquad = \psi \text{ if } ||Y||+m = ||X||+n,$

$\qquad = N(||X||+n-(||Y||+m)).$

$\breve{P}(n)P(m) = P(m-n),$

$\qquad = \Lambda \text{ if } m = n,$

$\qquad = \breve{P}(n-m).$

$\breve{P}(n)\breve{P}(m) = \breve{P}(n+m)$

$\breve{P}(n)\alpha(m) = \breve{P}(n+m)$

$\breve{P}(n)\breve{\alpha}(m) = P(m-n),$

$\qquad = \Lambda \text{ if } m = n,$

$\qquad = \breve{P}(n-m).$

$\breve{P}(n)\beta(m) = \breve{N}(m+||Y||-(||X||+n)),$

$\qquad = \psi \text{ if } m+||Y|| = ||X||+n,$

$\qquad = N(||X||+n-(||Y||+m)).$

$\breve{P}(n)\breve{\beta}(m) = \breve{N}(||Y||-(||X||+n+m)),$

$\qquad = \psi \text{ if } ||Y|| = ||X||+n+m,$

$\qquad = N(||X||+n+m-||Y||).$

$\breve{P}(n)\psi = \breve{N}(||Y||-(||X||+n)),$

$\qquad = \psi \text{ if } ||Y|| = ||X||+n,$

$\qquad = N(||X||+n-||Y||).$

$\breve{P}(n)\Lambda = \breve{P}(n)$

α(n)

$\alpha(n)\Sigma(i,j) = \alpha(n+j)$

$\alpha(n)\breve{\Sigma}(i,j) = \breve{P}(j-n)$

$\qquad\qquad\quad = \Lambda$ if $j = n$,

$\qquad\qquad\quad = P(n-j)$.

$\alpha(n)\Gamma(m) \quad = \Gamma(n+m)$

$\alpha(n)\breve{\Gamma}(m) \quad = P(||Y||-n+m)$

$\alpha(n)K \qquad = \Gamma(n)$

$\alpha(n)\breve{K} \qquad = P(||Y||-n)$

$\alpha(n)N(m) \quad = \breve{N}(||Y||-(||X||+n+m))$,

$\qquad\qquad\quad = \psi$ if $||Y|| = ||X||+n+m$,

$\qquad\qquad\quad = N(||X||+n+m-||Y||)$.

$\alpha(n)\breve{N}(m) \quad = \breve{N}(||Y||+m-(||X||+n))$,

$\qquad\qquad\quad = \psi$ if $||Y||+m = ||X||+n$,

$\qquad\qquad\quad = \breve{\beta}(||X||+n-(||Y||+m))$.

$\alpha(n)P(m) \quad = P(m-n)$,

$\qquad\qquad\quad = \Lambda$ if $m = n$,

$\qquad\qquad\quad = \breve{P}(n-m)$.

$\alpha(n)\breve{P}(m) \quad = \breve{P}(n+m)$

$\alpha(n)\alpha(m) \quad = \breve{P}(n+m)$

$\alpha(n)\breve{\alpha}(m) \quad = P(m-n)$,

$\qquad\qquad\quad = \Lambda$ if $n = m$,

$\qquad\qquad\quad = \breve{P}(n-m)$.

$\alpha(n)\beta(m) \quad = \breve{N}(m+||Y||-(n+||X||))$,

$\qquad\qquad\quad = \psi$ if $m+||Y|| = n+||X||$,

$\qquad\qquad\quad = \breve{\beta}(n+||X||-(m+||Y||))$.

$\alpha(n)\breve{\beta}(m) \quad = \breve{N}(||Y||-(||X||+n)-m)$,

$\qquad\qquad\quad = \psi$ if $||Y||-(||X||+n) = m$,

$\qquad\qquad\quad = N(m-(||Y||-(||X||+n)))$.

$\alpha(n)\psi = \breve{N}(||Y||-(||X||+n))$,

$\qquad\quad = \psi$ if $||Y|| = ||X||+n$,

$\qquad\quad = \breve{\beta}(||X||+n-||Y||)$.

$\alpha(n)\Lambda = \breve{P}(n)$

$\underline{\breve{\alpha}(n)}$ $\quad$ $\breve{\alpha}(n)\Sigma(i,j) = \alpha(j-n),$

$\qquad\qquad\qquad = \wedge$ if $j = n,$

$\qquad\qquad\qquad = \breve{\alpha}(n-j).$

$\breve{\alpha}(n)\breve{\Sigma}(i,j) = P(n+j)$

$\breve{\alpha}(n)\Gamma(m) \quad = N(||X||-n+m)$

$\breve{\alpha}(n)\breve{\Gamma}(m) \quad = P(m+||Y||+n)$

$\breve{\alpha}(n)K \qquad = \breve{\beta}(||X||-n)$

$\breve{\alpha}(n)\breve{K} \qquad = P(||Y||+n)$

$\breve{\alpha}(n)N(m) \qquad = \breve{N}(||Y||+n-||X||-m),$

$\qquad\qquad\qquad = \psi$ if $||Y||+n-||X|| = m,$

$\qquad\qquad\qquad = N(m-(||Y||+n-||X||)).$

$\breve{\alpha}(n)\breve{N}(m) \qquad = \breve{N}(m+||Y||+n-||X||),$

$\qquad\qquad\qquad = \psi$ if $m+||Y||+n = ||X||,$

$\qquad\qquad\qquad = \breve{\beta}(||X||-(m+||Y||+n)).$

$\breve{\alpha}(n)P(m) \qquad = P(n+m)$

$\breve{\alpha}(n)\breve{P}(m) \qquad = \breve{\alpha}(n-m),$

$\qquad\qquad\qquad = \wedge$ if $n = m,$

$\qquad\qquad\qquad = \breve{P}(m-n).$

$\breve{\alpha}(n)\alpha(m) \qquad = \breve{\alpha}(n-m),$

$\qquad\qquad\qquad = \wedge$ if $n = m,$

$\qquad\qquad\qquad = \alpha(m-n).$

$\breve{\alpha}(n)\breve{\alpha}(m) \qquad = P(n+m)$

$\breve{\alpha}(n)\beta(m) \qquad = \breve{N}(m+||Y||+n-||X||),$

$\qquad\qquad\qquad = \psi$ if $m+||Y||+n = ||X||,$

$\qquad\qquad\qquad = \breve{\beta}(||X||-(m+||Y||+n)).$

$\breve{\alpha}(n)\breve{\beta}(m) \qquad = \breve{N}(||Y||+n-(m+||X||)),$

$\qquad\qquad\qquad = \psi$ if $||Y||+n = m+||X||,$

$\qquad\qquad\qquad = \breve{\beta}(m+||X||-(||Y||+n)).$

$\breve{\alpha}(n)\psi = \breve{N}(||Y||+n-||X||),$

$\qquad\quad = \psi$ if $||Y||+n = ||X||,$

$\qquad\quad = \breve{\beta}(||X||-(||Y||+n)).$

$\breve{\alpha}(n)\wedge = \breve{\alpha}(n)$

$\underline{\beta(n)}$

$$\beta(n)\Sigma(i,j) = \tilde{N}(i+n)$$
$$\beta(n)\breve{\Sigma}(i,j) = \tilde{N}(n-i),$$
$$= \psi \text{ if } n = i,$$
$$= N(i-n).$$
$$\beta(n)\Gamma(m) = N(||Y||-n+m)$$
$$\beta(n)\breve{\Gamma}(m) = \breve{\Gamma}(n+m)$$
$$\beta(n)K = N(||Y||-n)$$
$$\beta(n)\breve{K} = \breve{\Gamma}(n)$$
$$\beta(n)N(m) = \tilde{N}(n-m),$$
$$= \psi \text{ if } n = m,$$
$$= N(m-n).$$
$$\beta(n)\tilde{N}(m) = \tilde{N}(n+m)$$
$$\beta(n)P(m) = \tilde{N}(n-(||Y||-(||Z||+m))),$$
$$= \psi \text{ if } n = ||Y||-(||Z||+m),$$
$$= N((||Y||-(||Z||+m))-n).$$
$$\beta(n)\breve{P}(m) = \beta(n-((||Y||+m)-||Z||)),$$
$$= \psi \text{ if } n = ||Z||+m-n,$$
$$= N(((||Y||+m)-||Z||)-n).$$
$$\beta(n)\alpha(m) = \tilde{\alpha}(n+||X||-(||Y||+m)),$$
$$= \Lambda \text{ if } n+||X|| = ||Y||+m,$$
$$= \breve{P}(||Y||+m-(||X||+n)).$$
$$\beta(n)\tilde{\alpha}(m) = \tilde{N}(||Z||+m+n-||Y||),$$
$$= \psi \text{ if } ||Z||+m+n = ||Y||,$$
$$= N(||Y||-(||Z||+m+n)).$$
$$\beta(n)\beta(m) = \tilde{N}(n+m)$$
$$\beta(n)\tilde{\beta}(m) = \tilde{N}(n-m),$$
$$= \psi \text{ if } n = m,$$
$$= N(m-n).$$

$$\beta(n)\psi = \tilde{N}(n)$$
$$\beta(n)\Lambda = \tilde{\alpha}(n+||X||-||Y||),$$
$$= \Lambda \text{ if } n+||X|| = ||Y||,$$
$$= \breve{P}(||Y||-(n+||X||)).$$

$\underline{\breve{\beta}(n)}$ $\breve{\beta}(n)\Sigma(i,j) = \beta(i\dot- n)$, $\qquad\qquad$ $\breve{\beta}(n)\psi = \breve{\beta}(n)$

$\qquad\qquad\qquad = \psi$ if $i = n$, $\qquad\qquad\qquad$ $\breve{\beta}(n)\Lambda = \breve{\alpha}(||X||\dot-(n+||Y||))$,

$\qquad\qquad\qquad = \breve{\beta}(n\dot- i)$. $\qquad\qquad\qquad\qquad = \Lambda$ if $||X|| = n+||Y||$,

$\breve{\beta}(n)\breve{\Sigma}(i,j) = N(n+i)$ $\qquad\qquad\qquad\qquad\quad = \breve{P}(n+||Y||\dot- ||X||)$.

$\breve{\beta}(n)\Gamma(m) \quad = N(n+||Y||+m)$

$\breve{\beta}(n)\breve{\Gamma}(m) \quad = P(||X||\dot- n+m)$

$\breve{\beta}(n)K \qquad = N(n+||Y||)$

$\breve{\beta}(n)\breve{K} \qquad = \breve{\alpha}(||X||\dot- n)$

$\breve{\beta}(n)N(m) \quad = N(n+m)$

$\breve{\beta}(n)\breve{N}(m) \quad = \breve{N}(m\dot- n)$,

$\qquad\qquad\qquad = \psi$ if $m = n$,

$\qquad\qquad\qquad = \breve{\beta}(n\dot- m)$.

$\breve{\beta}(n)P(m) \quad = P(m\dot-(||Y||+n-||X||))$,

$\qquad\qquad\qquad = \Lambda$ if $m = ||Y||+n\dot- ||X||$,

$\qquad\qquad\qquad = \breve{P}(||Y||+n\dot- ||X||\dot- m)$.

$\breve{\beta}(n)\breve{P}(m) \quad = \breve{\alpha}(||X||\dot-(n+||Y||+m))$,

$\qquad\qquad\qquad = \Lambda$ if $||X|| = n+||Y||+m$,

$\qquad\qquad\qquad = \breve{P}(n+||Y||+m\dot- ||Y||)$.

$\breve{\beta}(n)\alpha(m) \quad = \breve{\alpha}(||X||\dot-(n+||Y||+m))$,

$\qquad\qquad\qquad = \Lambda$ if $||X|| = n+||Y||+m$,

$\qquad\qquad\qquad = \breve{P}(n+||Y||+m\dot- ||X||)$.

$\breve{\beta}(n)\breve{\alpha}(m) \quad = \breve{\alpha}(m\dot-(||Y||+n\dot- ||X||))$,

$\qquad\qquad\qquad = \Lambda$ if $m = ||Y||+n\dot- ||X||$,

$\qquad\qquad\qquad = \breve{P}(||Y||+n\dot- ||X||\dot- m)$.

$\breve{\beta}(n)\beta(m) \quad = \beta(m\dot- n)$

$\qquad\qquad\qquad = \psi$ if $m = n$,

$\qquad\qquad\qquad = \breve{\beta}(n\dot- m)$.

$\breve{\beta}(n)\breve{\beta}(m) \quad = N(n+m)$

$\underline{\Psi}$ $\Psi\Sigma(i,j) = \widetilde{N}(i)$

$= \Psi$ if $i = 0$.

$\Psi\widetilde{\Sigma}(i,j) = N(i)$,

$= \Psi$ if $i = 0$.

$\Psi\Gamma(n) \quad = N(||Y||+m)$

$\Psi\widetilde{\Gamma}(n) \quad = \widetilde{\Gamma}(m)$

$\Psi K \qquad = N(||Y||)$

$\Psi\widetilde{K} \qquad = \widetilde{K}$

$\Psi N(n) \qquad = N(n)$

$\Psi\widetilde{N}(n) \qquad = \widetilde{N}(n)$

$\Psi P(n) \qquad = \widetilde{N}(||Z||+n-||Y||)$,

$= \Psi$ if $||Z||+n = ||Y||$,

$= N(||Y||-(||Z||+n))$.

$\Psi\widetilde{P}(n) \qquad = \widetilde{\alpha}(||X||-(||Y||+n))$,

$= \Lambda$ if $||X|| = ||Y||+n$,

$= \widetilde{P}(||Y||+n-||X||)$.

$\Psi\alpha(n) \qquad = \widetilde{\alpha}(||X||-(||Y||+n))$,

$= \Lambda$ if $||X|| = ||Y||+n$,

$= \widetilde{P}(||Y||+n-||X||)$.

$\Psi\widetilde{\alpha}(n) \qquad = \beta(||Z||+n-||Y||)$,

$= \Psi$ if $||Z||+n = ||Y||$,

$= N(||Y||-(||Z||+n))$.

$\Psi\beta(n) \qquad = \beta(n)$

$\Psi\widetilde{\beta}(n) \qquad = \dot{N}(n)$

$\Psi\Psi \qquad = \Psi$

$\Psi\Lambda \qquad = \beta(||Z||-||Y||)$,

$= \Psi$ if $||Z|| = ||Y||$,

$= N(||Y||-||Z||)$.

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$\underline{\Lambda}$ $\Lambda\Sigma(i,j) = \Lambda$ if $j = 0$,

                     $= \alpha(j)$.

$\Lambda\breve{\Sigma}(i,j) = \Lambda$ if $j = 0$,

                     $= \breve{\alpha}(j)$.

$\Lambda\,\Gamma(n)$ $= \Gamma(n)$

$\Lambda\,\breve{\Gamma}(n)$ $= P(n+||Y||)$

$\Lambda\,K$ $= K$

$\Lambda\,\breve{K}$ $= P(||Y||)$

$\Lambda\,N(n)$ $= P(||Y||-(n+||Z||))$,

                $= \Lambda$ if $||Y|| = n+||Z||$,

                $= \breve{P}(||Z||+n-||Y||)$.

$\Lambda\,\breve{N}(n)$ $= \breve{\beta}(||X||-(||Y||+n))$,

                $= \psi$ if $||X|| = ||Y||+n$,

                $= \breve{N}(||Y||+n-||X||)$.

$\Lambda\,P(n)$ $= P(n)$

$\Lambda\,\breve{P}(n)$ $= \breve{P}(n)$

$\Lambda\,\alpha(n)$ $= \alpha(n)$

$\Lambda\,\breve{\alpha}(n)$ $= P(n)$

$\Lambda\,\beta(n)$ $= \breve{\beta}(||X||-(||Y||+n))$,

                $= \psi$ if $||X|| = ||Y||+n$,

                $= \breve{N}(n+||Y||-||X||)$.

$\Lambda\,\breve{\beta}(n)$ $= \breve{\beta}(||X||+n-||Y||)$,

                $= \psi$ if $||X||+n = ||Y||$,

                $= \breve{N}(||Y||-(||X||+n))$.

$\Lambda\,\psi$ $= \breve{N}(||Y||-||X||)$,

                $= \psi$ if $||Y|| = ||X||$

                $= \breve{\beta}(||X||-||Y||)$.

$\Lambda\,\Lambda$ $= \Lambda$

APPENDIX III

METRIZED TRUTH TABLE

      The following table corresponds to the truth table of (Ref. 1),
with the exception that the entries for true ternary cycles are replaced
by equations which must hold among the m - variables of the cycles. The
general form of the cycle is taken as $X\omega(i)Y\overline{\omega}(j)X$. As in the case of
the metrized implication table relates are omitted. The relations Ω,
Ω', and Δ have not been included since the equations corresponding to
true cycles containing those relations are already represented in the
table (see pages 3 and 4). The table is symmetric and thus only
half of it is shown. If one encounters a blank for $X\omega(i)Y\overline{\omega}(j)X$, simply
form $X\overset{\smile}{\overline{\omega}}(j)Y\overset{\smile}{\omega}(i)X$ to find the corresponding equation.

Metrized Truth Table

	Φ	Σ(n,m)	Σ(n,m)	Γ(n)	Γ(n)	K	K	N(n)	N(n)	N(n)	P(n)	P(n)	σ(n)	α(n)	ξ(n)	ξ(n)	τ	τ
φ	$\|x\|=\|y\|$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Σ(i,j)	0	0	$i+\|x\|+j=\|y\|$	0	0	0	0	0	$n+\|x\|=\|y\|$	$i+\|x\|+n=\|y\|$	0	$i+\|x\|+n=\|y\|$	0	$i+\|x\|+n=\|y\|$	0	$n+\|x\|+j=\|y\|$	$\|x\|+j=\|y\|$	$i+\|x\|+j=\|y\|$
Σ(i,j)	0	0	0	0	0	0	0	0	0	$i+\|x\|=\|y\|$	$i+\|x\|=\|y\|$	0	0	0	0	0	$\|y\|+j=\|x\|$	$i+\|x\|=\|y\|$
Γ(i)	0	0	0	$i=n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Γ(i)	0	0	0	0	0	0	0	$i+\|y\|=n$	$i+\|x\|=n$	0	$i+\|x\|=n$	0	0	0	0	0	0	$i+\|y\|=\|x\|$
K	0	0	0	0	0	0	X	0	0	0	0	0	0	0	0	0	0	0
K	0	0	0	0	0	0	0	$\|x\|=n$	0	$n=\|x\|$	0	$n=\|x\|$	0	0	0	0	0	0
N(i)	0	0	0	0	0			0	$i=n$	$i+\|x\|+n=i+\|y\|$	$i+\|y\|+n=\|x\|$	$i+\|x\|+n=\|y\|$	$i=n$	0	0	$i=n$	0	$i+\|x\|=\|y\|$
N(i)	0	0	0	0	0			0		$i+\|x\|+n=\|y\|$	$i+\|x\|+n=\|y\|$		0	$i=n$	$i=n$	0	0	$i+\|x\|=\|y\|$
P(i)	0									0	$i+\|y\|+n=\|x\|$	$i=n$	0	$i+\|y\|+n=\|x\|$	$i+\|y\|+n=\|x\|$	$i+\|y\|+n=\|x\|$	0	0
P(i)	0									$i=n$	0	0	$i=n$	0	$i+\|x\|+n=\|y\|$	$i+\|x\|=\|y\|$	0	0
α(i)											$i=n$		0	$i=n$	$i+\|x\|+n=\|y\|$	$i+\|x\|+n=\|y\|$	0	0
α(i)											0		0		$i+\|y\|=\|x\|$	$i+\|y\|+n=\|x\|$	0	0
θ(i)															0	$i=n$	$i+\|x\|=\|y\|$	$i+\|x\|=\|y\|$
θ(i)																$i=n$	$i+\|y\|=\|x\|$	$i+\|y\|=\|x\|$
ψ																	X	X

Metrized Truth Table

(X signifies that the cycle is true, but no equation exists for it).

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    Constraints, (Project 611.1), J. F. Rial, TM-04271, The MITRE
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    The MITRE Corporation, Bedford, Massachusetts.

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13. ABSTRACT

The definitions of scheduling relations previously presented by the author are extended in this report to allow the translation of a relation net into an exact schedule. Time conditional conflicts in scheduling relation nets are detected and resolved by operating on extended versions of the bilateral implication and truth tables.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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